

# Critical Configurations for $N$ -view Projective Reconstruction

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## Abstract

*In this paper we give a characterization of critical configurations for projective reconstruction with any number of points and views. A set of cameras and points is said to be critical if the projected image points are insufficient to determine the placement of the points and the cameras uniquely, up to a projective transformation. For two views, the critical configurations are well-known. In this paper it is shown that a configuration of  $n \geq 3$  cameras and  $m$  points all lying on the intersection of two distinct ruled quadrics is critical. In distinction to the two-view case, which in general allows two alternative solutions, there is a family of ambiguous reconstructions for the  $n$ -view case. As a partial converse, it is shown that for any critical configuration, all the points lie on the intersection of two ruled quadrics.*

## 1 Introduction

A key problem in computer vision is to recover the shape of an object from a number of images. This inverse problem has a number of inherent ambiguities. It is well-known that from image measurements alone, the cameras and the 3D points can only be determined up to an unknown projective transformation. For two views, additional ambiguities occur if all points and cameras lie on a ruled quadric. This critical surface or “gefährlicher Ort” was studied by Krames [7] in 1940. See [8, 3] for a more recent treatment.

In this paper, we consider the problem of ambiguity of projective reconstruction from three or more views. Understanding of the two-view case will be helpful for reading this paper, though the important prerequisite results will be quoted here for convenience. We follow the approach and terminology used in [2, 3]. Other work on critical curves and surfaces can be found in [5, 1, 6]. Partial results concerning ambiguous configurations with more than two views have been reported previously in the literature. Maybank and Shashua ([9]) considered the case of many views of 6 points, showing that a configuration is critical if and only if the points and camera all lie on a quadric.

The first non-trivial examples of critical configurations for three views were given in [2] in which it was shown that three cameras always belong to some critical configuration of points in which the cameras and points all lie on the in-

tersection of two ruled quadrics, known as an elliptic quartic curve<sup>1</sup>. Furthermore in *any* critical configuration, all points (but not necessarily the cameras) must lie on the intersection of three (and hence two) ruled quadrics. However, the example given in [2] was somewhat special, and the full range of critical curves was not given. In the present paper, we show that *any* curve formed as the intersection of two ruled quadrics is critical – a set of cameras and points all lying on this intersection curve allow an ambiguous reconstruction. This is a much more general result than the single example of a critical curve given in [2].

The result is extended by showing that the same quartic curve is a critical set for  $n$ -view reconstruction. Thus, one may add any number of further cameras with centres located on the critical curve without removing the reconstruction ambiguity – the curve remains critical for all the views.

## 2 Statement of the problem

We consider a configuration of  $n \geq 3$  cameras, and denote the camera matrices by  $\mathbf{P}^i$ , for  $i = 0, \dots, n-1$ . Consider also a set of points  $\mathbf{P}_j$ . The question considered is under what circumstances there exists an alternative set of camera matrices  $\mathbf{Q}^i$  and points  $\mathbf{Q}_j$  such that  $\mathbf{P}^i \mathbf{P}_j = \mathbf{Q}^i \mathbf{Q}_j$  for all  $i, j$ , but  $\{\mathbf{P}_j\}$  and  $\{\mathbf{Q}_j\}$  are projectively inequivalent point sets. If such an alternative set of points and cameras exist, then we say that the configuration  $\{\mathbf{P}^i, \mathbf{P}_j\}$  is a *critical configuration*. In this case, the alternative configuration  $\{\mathbf{Q}^i, \mathbf{Q}_j\}$  is its *conjugate configuration*.<sup>2</sup>

**Standard camera configurations.** It is useful to put the three camera matrices in a canonical form. The first remark is that the precise form of the camera matrices  $\mathbf{P}^i$  is not important for criticality, just their centres. This is proved in [2] (Proposition 1). For convenience, it may therefore be assumed that the three camera matrices are of the form  $\mathbf{P}^i = [\mathbf{I} \mid -\mathbf{v}^i]$ , where  $\mathbf{v}^i$  is the camera centre.

<sup>1</sup>In the classification of space curves [10], there are two types of irreducible quartics, elliptic quartics (the intersection of two quadrics) and rational quartics (the intersection of a cubic and a quadric minus two lines).

<sup>2</sup>This definition has a slight technical difficulty in that the location of points lying on the line between two camera centres is not determined by their projection, but may vary arbitrarily along the base line. We are not interested in this trivial sort of ambiguity.

**Critical surfaces for pairs of cameras.** If  $\{\mathbf{P}^0, \mathbf{P}^1, \mathbf{P}^2, \mathbf{P}_k\}$  is a critical configuration for three views, then it is critical for each of the three pairs of cameras. Therefore, it follows from the well-known results on 2-view critical surfaces that the set of points  $\mathbf{P}_k$  and the camera centres of  $\mathbf{P}^i$  and  $\mathbf{P}^j$  lie on a ruled quadric surface, denoted  $S_p^{ij}$ . According to [2], the quadric is given by the formula

$$S_p^{ij} = \mathbf{P}^i \top \mathbf{F}_Q^{ij} \mathbf{P}^j \quad (1)$$

where  $\mathbf{F}_Q^{ij}$  is the fundamental matrix for the pair of cameras  $(Q^i, Q^j)$  in the conjugate configuration<sup>3</sup>. It is also shown in [2] that when  $S_p^{10}, S_p^{20}$  and  $S_p^{21}$  are equal, the corresponding ruled quadric is not critical.

### 3 Pencils of quadrics

Given two quadrics represented by symmetric  $4 \times 4$  matrices  $A$  and  $B$ , the pencil generated by them consists of all quadrics of the form  $\alpha A + \beta B$ .

**Proposition 3.1.** *If  $C$  is the intersection of the quadrics  $A$  and  $B$ , then  $C$  lies on each of the quadrics  $\alpha A + \beta B$  in the pencil. If*

$$\det \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \neq 0$$

*then the intersection of quadrics  $\alpha A + \beta B$  and  $\gamma A + \delta B$  is the same as the intersection of the quadrics  $A$  and  $B$ .*

A pencil of quadrics may be defined by either ruled or unruled quadrics. It is easily verified by example that a pencil of quadrics defined by two non-ruled quadrics may contain ruled quadrics, and even imaginary quadrics (those with no real points). Similarly, a pencil defined by ruled quadrics may contain non-ruled quadrics. However, it is easily seen that a pencil that contains a non-degenerate ruled quadric (a hyperboloid of one sheet) is generated by two such non-degenerate ruled quadrics. The intersection of two quadrics is a fourth-degree curve called an *elliptic quartic* [10]. It has 16 degrees of freedom ( $2 \times 9 - 2 = 16$ , two quadrics minus the choice of base quadrics defining the pencil). Thus, 8 points in general position define an elliptic quartic uniquely.

### 4 Critical surfaces for 3 views

Now, we prove the main theorem concerning critical configurations in three views. The generalization to  $n$  views and its converse will be considered in the proceeding sections.

**Theorem 4.2.** *A configuration of three cameras  $\mathbf{P}^i$  and points  $\mathbf{P}_j$  lying on the intersection of two distinct non-degenerate ruled quadrics is critical.*

<sup>3</sup>It is common practice to represent quadrics by *symmetric*  $4 \times 4$  matrices, but in this paper we represent them by non-symmetric matrices such as the  $S_p^{ij}$  defined here.

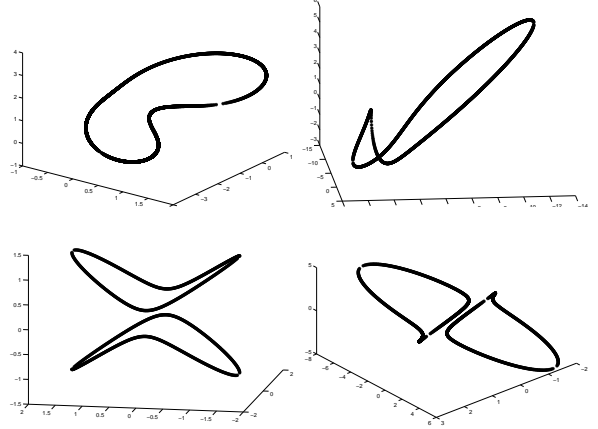


Figure 1: *Examples of randomly generated elliptic quartics. In all examples, the pencils contain ruled quadrics.*

**Proof.** The proof consists of exhibiting an explicit formula for the conjugate configurations. Without loss of generality, we may assume that  $S_p^{10}$  is the quadric  $z = xy$ , represented by the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad (2)$$

since all hyperboloids of one sheet are projectively equivalent. Further, we may assume that the camera centres are the points  $(0, 0, 0)^\top$  and  $(1, 1, 1)^\top$  and  $(-1, -1, 1)^\top$ .<sup>4</sup> The camera matrices may then be taken as  $\mathbf{P}^0 = [\mathbf{I} | 0]$  and  $\mathbf{P}^1 = [\mathbf{I} | (-1, -1, -1)^\top]$  and  $\mathbf{P}^2 = [\mathbf{I} | (1, 1, -1)^\top]$ .

Next, we want another quadric  $B$  that contains the three camera centres. As  $B$  varies over all such quadrics, the intersection of  $A$  and  $B$  encompasses all elliptic quartics passing through the three camera centres. The condition that the given camera centres lie on  $B$  implies three linear restrictions on the entries of  $B$ . In addition, it may be assumed that  $B_{34} = B_{43} = 0$ , since this may otherwise be achieved by adding a suitable multiple of  $A$ , without changing the intersection of the two quadrics. Under these conditions, it is easy to verify that  $B$  is of the form

$$\begin{bmatrix} 2p & q & s-t & -s-u \\ q & 2r & s+t & -s+u \\ s-t & s+t & -2(p+q+r) & 0 \\ -s-u & -s+u & 0 & 0 \end{bmatrix}. \quad (3)$$

Thus, the matrix  $B$  is defined by 6 parameters,  $\{p, q, r, s, t, u\}$ .

<sup>4</sup>Assuming that the three camera centres do not lie on the same generator of the quadric. If the three centres are on the same generator, an analogous proof can be derived.

Two alternative reconstructions involving cameras  $\mathbf{Q}^i$  and points  $\mathbf{Q}$  are given in Table 1 and Table 2. It may be verified directly that  $\mathbf{P}^i \mathbf{P} = \mathbf{Q}^i \mathbf{Q}$  for all points  $\mathbf{P} = (x, y, xy, 1)^\top$  and corresponding points  $\mathbf{Q}$ , provided that  $\mathbf{P}$  lies on the quadric  $B$ . (It always lies on quadric  $A$ ). The easiest way to see this is to verify that  $(\mathbf{P}^i \mathbf{P}) \times (\mathbf{Q}^i \mathbf{Q}) = 0$  for all such points. In fact for  $i = 0, 1$ , the cross-product is always zero, whereas for  $i = 2$  it may be verified by direct computation that

$$(\mathbf{P}^2 \mathbf{P}) \times (\mathbf{Q}^2 \mathbf{Q}) = (\mathbf{P}^\top \mathbf{B} \mathbf{P}) (4, -4x, 4)^\top$$

for the first solution, and

$$(\mathbf{P}^2 \mathbf{P}) \times (\mathbf{Q}^2 \mathbf{Q}) = (\mathbf{P}^\top \mathbf{B} \mathbf{P}) (-4y, 4, 4)^\top$$

for the second solution. Thus  $\mathbf{P}^2 \mathbf{P} = \mathbf{Q}^2 \mathbf{Q}$  if and only if  $\mathbf{P}$  lies on  $B$ . ■

## 5 A family of solutions

Given a quartic curve defined as the intersection of a pencil of quadrics spanned by  $A$  and  $B$ , Table 1 and Table 2 give examples of conjugate solutions for which the critical quadric  $S_p^{10}$  is equal to  $A$ . However,  $A$  is just one of a family of quadrics that may be used to span the pencil. For all values of a parameter  $\gamma$ , the quadric  $\gamma A + B$  may be chosen as one of such a pair of spanning quadrics. Provided  $\gamma A + B$  is a ruled quadric, the argument and examples of section 4 show that there is a conjugate configuration for which  $S_p^{10} = \gamma A + B$ .

It is easy to see that the conjugate solutions arising from two different quadrics  $A$  and  $A'$  are projectively inequivalent. For let  $\mathbf{F}_q^{10}$  and  $\mathbf{F}_q'^{10}$  be the fundamental matrices for the two different configurations. Since  $A = \mathbf{P}^{1\top} \mathbf{F}_q^{10} \mathbf{P}^0$  and  $A' = \mathbf{P}^{1\top} \mathbf{F}_q'^{10} \mathbf{P}^0$  are different, it follows that  $\mathbf{F}_q^{10} \neq \mathbf{F}_q'^{10}$ . Since the fundamental matrices are different, the two configurations are not projectively equivalent. This shows

**Theorem 5.3.** *Suppose three cameras, and a set of points  $\{\mathbf{P}^i, \mathbf{P}_j\}$  lie on a curve defined as the intersection of a pencil of quadrics. Then for each non-degenerate ruled quadric  $A$  in the pencil, there exists a pair of conjugate configurations, each of the form  $\{\mathbf{Q}^i, \mathbf{Q}_j\}$ , such that the critical quadric  $S_p^{10} = \mathbf{P}^{1\top} \mathbf{F}_q^{10} \mathbf{P}^0 = A$ .*

Thus, unlike critical configurations in the two-view case, which allow two conjugate solutions, critical configurations for three views give rise to two one-parameter families of conjugate configurations.

## 6 Linear mapping

Since the conjugate solutions were seemingly pulled out of a hat, we will give a more theoretical treatment of the problem now, which will partly elucidate the method used to arrive at this solution.

We start by giving some properties of critical quadrics for two views, as given by (1). For the proofs, see [2].

**Theorem 6.4.** *Given camera matrices  $\mathbf{P}$  and  $\mathbf{P}'$  and a  $3 \times 3$  matrix  $\mathbf{F}$ , define  $S = \mathbf{P}'^\top \mathbf{F} \mathbf{P}$ . Let  $S_{\text{sym}} = S + S^\top$  represent a quadric surface. Then (i)  $S_{\text{sym}}$  is zero if and only if  $\mathbf{F}$  is the fundamental matrix corresponding to the pair  $(\mathbf{P}, \mathbf{P}')$ . (ii) If non-zero, then  $S_{\text{sym}}$  represents a ruled quadric provided  $\det \mathbf{F} = 0$ . (iii) The camera centres of both  $\mathbf{P}$  and  $\mathbf{P}'$  lie on  $S_{\text{sym}}$ .*

Note the important information that this construction defines a ruled quadric, provided  $\det \mathbf{F} = 0$ .

Let  $\mathbf{P}$  and  $\mathbf{P}'$  be fixed, and consider the mapping  $f : \mathbf{F} \mapsto S_{\text{sym}}$  defined in Theorem 6.4. Let us count the dimensions of the range and domain of this mapping. The domain of  $f$  is the 9-dimensional space consisting of all  $3 \times 3$  matrices, effectively  $\mathcal{R}^9$ . The dimension of the space of all  $4 \times 4$  symmetric matrices is 10. However, the constraint that the quadric  $S_{\text{sym}}$  should pass through a given point (a camera centre) gives a single linear constraint on the entries of  $S_{\text{sym}}$ . Hence, the set of symmetric matrices representing quadrics passing through two given points has dimension 8, and is in fact an 8-dimensional subspace of  $\mathcal{R}^{10}$ . Thus,  $f$  is a linear mapping from  $\mathcal{R}^9$  to  $\mathcal{R}^8$ . According to Theorem 6.4, the mapping  $f$  has a 1-dimensional kernel. It follows that  $f$  is an epimorphism (onto-mapping).

This result does not take into account the condition  $\det \mathbf{F} = 0$  for a matrix to be a fundamental matrix. The restriction of this map to the set of all zero-determinant fundamental matrices defines a mapping from the set of all fundamental matrices to a set of ruled quadrics, as enunciated in the next theorem.

**Theorem 6.5.** *Let  $\mathbf{P}$  and  $\mathbf{P}'$  be fixed and define  $f(\mathbf{F}) = (\mathbf{P}'^\top \mathbf{F} \mathbf{P})_{\text{sym}}$ . If  $\mathbf{F}$  has zero determinant (hence is a fundamental matrix), then  $f(\mathbf{F})$  represents a ruled quadric passing through the camera centres of  $\mathbf{P}$  and  $\mathbf{P}'$ . Conversely, let  $S$  be any non-degenerate ruled quadric passing through the centres of  $\mathbf{P}$  and  $\mathbf{P}'$ . Then there exists at least one fundamental matrix  $\mathbf{F}$  such that  $f(\mathbf{F}) = S$ .*

**Proof.** This will be shown for the case where the two camera centres lie on different generators of the quadric. Without loss of generality, we may assume that  $S$  is the quadric  $z = xy$ , since all hyperboloids of one sheet are projectively equivalent. Further, we may assume that the camera matrices are of the form  $\mathbf{P} = \mathbf{K}[\mathbf{I}|0]$  and  $\mathbf{P}' = \mathbf{K}'[\mathbf{I} | (1, 1, 1)^\top]$ , where  $\mathbf{K}$  and  $\mathbf{K}'$  are non-singular matrices.

Now, it may be verified easily (see the example on page 929 of [2]) that the matrix  $\mathbf{F} = \mathbf{K}'^{-\top} \mathbf{G} \mathbf{K}^{-1}$  gives the required result, where

$$\mathbf{G} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad \text{or} \quad \mathbf{G} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

The camera matrices are	$\mathbf{Q}^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{Q}^1 = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -2 & 1 \end{bmatrix}$
and	$\mathbf{Q}^2 = \begin{bmatrix} -4(2p+q-t+u) & 8r & 4(p+q+2r+s+t) & -2(p+q-s-t) \\ 0 & 8(r+s-u) & -2(q-t+u) & -q+t-u \\ 8p & -8r & -2(2p+q-2s+3t+3u) & 2p+q-2s-t-u \end{bmatrix}.$
The conjugate point to $\mathbf{P} = (x, y, xy, 1)^\top$ is $\mathbf{Q} = ((x-1)x, (x-1)y, (x-1)xy, -2x(-2+y+xy))^\top$ .	

Table 1: First conjugate solution to the reconstruction problem given by the quadrics  $A$  and  $B$  in (2) and (3), respectively.

The camera matrices are	$\mathbf{Q}^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{Q}^1 = \begin{bmatrix} 0 & 0 & -2 & 1 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$
and	$\mathbf{Q}^2 = \begin{bmatrix} -8(p+s+u) & 0 & 2(q+t-u) & -q-t+u \\ -8p & 4(q+2r+t-u) & -4(2p+q+r+s-t) & -2(q+r-s+t) \\ 8p & -8r & 2(q+2r-2s-3t-3u) & q+2r-2s+t+u \end{bmatrix}.$
The conjugate point to $\mathbf{P} = (x, y, xy, 1)^\top$ is $\mathbf{Q} = ((y-1)x, (y-1)y, (y-1)xy, 2y(-2+x+xy))^\top$ .	

Table 2: Second conjugate solution to the reconstruction problem.

Thus, there are two separate choices of fundamental matrix  $\mathbf{F}$  that map onto the ruled quadric  $S$ .

The case where the two camera centres lie on the same generator is treated similarly, but there exists only one  $\mathbf{F}$  in this case. ■

## 7 Compatible fundamental matrices

We return to the consideration of three camera matrices  $\mathbf{P}^i; i = 0, \dots, 2$ . For each pair of camera matrices,  $\mathbf{P}^i$  and  $\mathbf{P}^j$ , define the linear mapping  $f : \mathbf{F} \mapsto S_{\text{sym}}$  as described in Theorem 6.4. The three mappings obtained in this way may be denoted by  $f^{ij}$ , namely  $f^{10}$ ,  $f^{20}$  and  $f^{21}$ . Given three fundamental matrices  $\mathbf{F}_q^{ij}$ , we can form the three quadrics  $f^{ij}(\mathbf{F}_q^{ij})$ .

In this context, the three fundamental matrices  $\mathbf{F}_q^{ij}$  are the fundamental matrices corresponding to pairs of views in a conjugate configuration involving camera matrices  $\mathbf{Q}^i; i = 0, \dots, 2$ . An essential issue here is the *compatibility* of the three fundamental matrices.

**Definition 7.6.** Three fundamental matrices  $\mathbf{F}^{01}$ ,  $\mathbf{F}^{02}$  and  $\mathbf{F}^{12}$  are *compatible* if they satisfy the conditions

$$\mathbf{e}^{12\top} \mathbf{F}^{10} \mathbf{e}^{02} = 0, \quad \mathbf{e}^{21\top} \mathbf{F}^{20} \mathbf{e}^{01} = 0, \quad \mathbf{e}^{20\top} \mathbf{F}^{21} \mathbf{e}^{10} = 0.$$

The importance of this condition is given by the following theorem [4].

**Theorem 7.7.** There exist three camera matrices  $\mathbf{Q}^0, \mathbf{Q}^1, \mathbf{Q}^2$  such that  $\mathbf{F}^{ij}$  is the fundamental matrix corresponding to the pair  $(\mathbf{Q}^i, \mathbf{Q}^j)$  only if the three fundamental matrices  $\mathbf{F}^{10}$ ,  $\mathbf{F}^{20}$  and  $\mathbf{F}^{21}$  are compatible.

The converse of this theorem is probably true as well, but no complete proof seems to have appeared. For our present purposes the given statement is sufficient.

## 8 Discovering the conjugate solutions

We suppose given a pair of ruled quadrics  $A$  and  $B$  which intersect in a space curve. Without loss of ambiguity we can assume that the quadrics are the ones given by (2) and (3). Contained on this curve are the centres of three cameras  $\mathbf{P}^0$ ,  $\mathbf{P}^1$  and  $\mathbf{P}^2$ , as well as any number of points  $\mathbf{P}_j$ . If this is to be a critical configuration, then there will exist three conjugate cameras  $\mathbf{Q}^0, \mathbf{Q}^1$  and  $\mathbf{Q}^2$  and also points  $\mathbf{Q}_j$ . The three-view critical configuration will also be critical as a two-view configuration for each of the pairs of cameras, and hence there will exist three critical quadrics  $S_p^{10}$ ,  $S_p^{20}$  and  $S_p^{21}$  defined by (1), all meeting along the curve defined by  $A$  and  $B$ . The task then is to find three fundamental matrices  $\mathbf{F}_q^{ij}$ , which must satisfy the following conditions.

1. The three quadrics  $S_p^{ij}$  defined by (1) in terms of the given  $\mathbf{F}_q^{ij}$  must belong to the pencil of quadrics defined by  $A$  and  $B$ .

2. The three fundamental matrices  $\mathbf{F}_q^{ij}$  must be compatible in the sense of definition 7.6.

If these conditions hold, then it is generally possible to extract three camera matrices  $\mathbf{Q}^i$  that give rise to the three fundamental matrices  $\mathbf{F}_q^{ij}$ .

Consider the condition that  $S_p^{ij}$  must belong to the pencil defined by  $A$  and  $B$ . We assume that

$$S_p^{10} = A, \quad S_p^{20} = \alpha_0 A + B \quad \text{and} \quad S_p^{21} = \alpha_1 A + B.$$

Now, according to Theorem 6.5 there exist two choices of fundamental matrix  $\mathbf{F}_q^{10}$  such that  $f^{10}(\mathbf{F}_q^{10}) = A$ . We select one of them – in fact the two different conjugate configurations given in Table 1 and Table 2 correspond to the choice of the two possible  $\mathbf{F}_q^{10}$  at this stage. Now, for  $(i, j) = (2, 0)$  or  $(2, 1)$ , find matrices  $\mathbf{G}^{ij}$  and  $\mathbf{H}^{ij}$  (not necessarily singular) such that

$$f^{ij}(\mathbf{G}^{ij}) = A, \quad f^{ij}(\mathbf{H}^{ij}) = B.$$

Now, set

$$\begin{aligned} \mathbf{F}_q^{20} &= \alpha_0 \mathbf{G}^{20} + \mathbf{H}^{20} + \beta_0 \mathbf{F}_p^{20}, \\ \mathbf{F}_q^{21} &= \alpha_1 \mathbf{G}^{21} + \mathbf{H}^{21} + \beta_1 \mathbf{F}_p^{21}. \end{aligned}$$

Since the mapping  $f^{ij}$  is linear, and maps  $\mathbf{F}_p^{ij}$  to zero, it follows that

$$f^{ij}(\mathbf{F}_q^{ij}) = \alpha_j f^{ij}(\mathbf{G}^{ij}) + f^{ij}(\mathbf{H}^{ij}) = \alpha_j A + B = S_p^{ij}.$$

Unfortunately, the  $\mathbf{F}_q^{ij}$  arbitrarily chosen in this way are neither singular matrices nor compatible. We need to choose  $\alpha_j$  and  $\beta_j$  (four parameters) to satisfy the constraints  $\det(\mathbf{F}_q^{ij}) = 0$ , and the three compatibility constraints of definition 7.6. Note that  $\mathbf{F}_q^{10}$  was chosen such that  $\det(\mathbf{F}_q^{10}) = 0$ , so there are only five remaining constraints.

We are faced with the problem of satisfying five non-linear constraints with only four parameters. In addition, the constraints are non-linear – the determinant constraints are cubic, whereas the compatibility constraints involve extracting epipoles (null-spaces of the matrices). No symbolic algebra package (Mathematica or Maple) seems capable of solving this system without help even for numerical examples, and let alone the general case where  $B$  is a function of symbolic parameters, as in (3). Clearly we are going to need luck!

Since  $A$  may be assumed to be of a simple form (2), it is easy to compute  $\mathbf{F}_q^{10}$ , and from it extract the epipoles  $\mathbf{e}^{10}$  and  $\mathbf{e}^{01}$ .<sup>5</sup> Now, one of the compatibility conditions is  $\mathbf{e}^{21\top}(\mathbf{F}_q^{20} \mathbf{e}^{01}) = 0$ , which may be thought of as an equation involving the epipole  $\mathbf{e}^{21}$ . The equation  $\mathbf{e}^{21\top} \mathbf{F}_q^{21} = 0$  gives three further equations involving the epipole  $\mathbf{e}^{21}$ , which also ensure that  $\mathbf{F}_q^{21}$  has zero determinant. Numerical examples

<sup>5</sup>Epipoles are labelled according to the convention that  $\mathbf{e}^{ij\top} \mathbf{F}_q^{ij} = 0$ .

suggested that there were solutions to this set of equations independent of the particular form of  $\mathbf{F}_q^{20}$ , that is, valid for all values of  $\alpha_0$  and  $\beta_0$ . This would imply that for some values of  $(\alpha_1, \beta_1)$  there is a solution for  $\mathbf{e}^{21}$  to the set of equations

$$\begin{aligned} \mathbf{e}^{21\top}(\mathbf{G}^{20} \mathbf{e}^{01}) &= \mathbf{e}^{21\top}(\mathbf{H}^{20} \mathbf{e}^{01}) = \mathbf{e}^{21\top}(\mathbf{F}_p^{20} \mathbf{e}^{01}) = 0 \\ \mathbf{e}^{21\top} \mathbf{F}_q^{21} &= \mathbf{0}. \end{aligned}$$

Note that the first set of three equations does not involve any of the variables  $\alpha_j$  or  $\beta_j$ , since  $\mathbf{G}^{20}, \mathbf{H}^{20}, \mathbf{F}_p^{20}$  are known quantities (dependent only on the parameters of  $A$  and  $B$ ). The final equation (actually three equations) is linear in  $\alpha_1$  and  $\beta_1$ . If such a solution for  $\mathbf{e}^{21}$  exists, then a solution for  $(\alpha_1, \beta_1)$  can be obtained as follows:

1. Find the value of  $\mathbf{e}^{21}$  as the left null-space of the matrix  $[\mathbf{G}^{20} \mathbf{e}^{01}, \mathbf{H}^{20} \mathbf{e}^{01}, \mathbf{F}_p^{20} \mathbf{e}^{01}]$ .
2. Linearly solve the equation  $\mathbf{e}^{21\top} \mathbf{F}_q^{21} = \mathbf{0}$  for  $(\alpha_1, \beta_1)$ .

Thus, the solution for  $(\alpha_1, \beta_1)$  is obtained by solving only linear equations, which explains the absence of radicals in the solution given in Table 1 and Table 2. A similar procedure may be used to solve for the remaining variables  $(\alpha_0, \beta_0)$  defining fundamental matrix  $\mathbf{F}_q^{20}$ . The epipoles are next extracted from the computed  $\mathbf{F}_q^{20}$  and  $\mathbf{F}_q^{21}$  and the third consistency condition  $\mathbf{e}^{12\top} \mathbf{F}_q^{10} \mathbf{e}^{02} = 0$  verified. Finally, the camera matrices  $\mathbf{Q}^i$  are extracted from the three consistent fundamental matrices. Necessary simplifications of the obtained matrices are done interactively.

This whole computation was carried out for a general symbolic value of  $B$  given by (3), resulting in the general solution given in Table 1 and Table 2, and thereby justifying the procedure.

## 9 $N$ -view critical configurations

We now wish to extend this result to  $n$  views. By adding a fourth camera to a critical three-view configuration, one might suspect that the criticality would disappear. However, this is not the case if the additional camera lies on the same critical curve.

The goal of this section is the following theorem.

**Theorem 9.8.** *A configuration of  $n \geq 3$  cameras  $\mathbf{P}^i, i = 0, \dots, n-1$  and points  $\mathbf{P}_j$  all lying on the intersection of two distinct non-degenerate ruled quadrics is critical.*

**Proof.** Assume that the configuration lies on the intersection of two quadrics  $A$  and  $B$ . For simplicity of notation, we consider the 4-view case. The result for  $n$  views follows by induction. The three cameras  $\mathbf{P}^0, \mathbf{P}^1, \mathbf{P}^2$  along with the points form a critical configuration, and hence a conjugate configuration exists. Similarly a conjugate configuration exists for the points along with the three cameras  $\mathbf{P}^0, \mathbf{P}^1, \mathbf{P}^3$ . The goal

is to show that these two conjugate configurations are consistent.

Consider the way the cameras  $\mathbf{Q}^0$  and  $\mathbf{Q}^1$  are constructed. Let  $\mathbf{P}^i; i = 0, \dots, 2$  be the first triple of cameras. According to Theorem 5.3 they and the points  $\mathbf{P}_j$  lie (without loss of generality) on the quadric  $A = S_{\mathbf{P}}^{10}$ . The fundamental matrix  $F_{\mathbf{Q}}^{10}$  is chosen so that  $A = f^{10}(F_{\mathbf{Q}}^{10})$ . From this value of  $F_{\mathbf{Q}}^{10}$  two camera matrices  $\mathbf{Q}^0$  and  $\mathbf{Q}^1$  may be computed. The choice of the pair  $(\mathbf{Q}^0, \mathbf{Q}^1)$  is unique up to a projectivity.

We proceed in the same way with the second triple of cameras  $\mathbf{P}^0, \mathbf{P}^1, \mathbf{P}^3$ . Since the camera  $\mathbf{P}^2$  or  $\mathbf{P}^3$  does not take part in this construction, the resulting fundamental matrix  $F_{\mathbf{Q}}^{10}$  is the same, and from it one may extract two camera matrices, which will be denoted  $(\mathbf{Q}^{10}, \mathbf{Q}^{11})$ , which must be projectively equivalent to the pair  $(\mathbf{Q}^0, \mathbf{Q}^1)$ . However (and this is the main point) a conjugate configuration is defined only up to projectivity. Consequently, it is possible to choose the camera pair  $(\mathbf{Q}^{10}, \mathbf{Q}^{11})$  to be identical to  $(\mathbf{Q}^0, \mathbf{Q}^1)$ . In this way, from the two conjugate configurations corresponding to the two triples of cameras with indices  $(0, 1, 2)$  and  $(0, 1, 3)$ , we get four camera matrices  $\mathbf{Q}^i; i = 0, \dots, 3$ . Since the image of a point in two views is sufficient to determine its position (by triangulation), it follows that  $\mathbf{Q}_j = \mathbf{Q}'_j$ . ■

As in Theorem 5.3 for the three-view case, there exists a family of conjugate configurations in the  $n$ -view case also.

## 10 Converse

It is natural to ask whether the critical configurations considered here account for all possible critical configurations involving  $n \geq 3$  cameras. The answer is no. There do exist other configurations, which will be discussed in future papers. However, in a sense to be made clear below, most critical configurations are of the type discussed in this paper. In [2] it was shown that all the points in a critical configuration must lie in the intersection of all three quadrics  $S_{\mathbf{P}}^{ij}$ ; but this does not hold for the camera centres.

According to Theorem 6.4 and [2], the centre of the camera  $\mathbf{P}^i$  belonging to a critical configuration lies on each of the quadrics  $S_{\mathbf{P}}^{ik}$  for all  $k$ , as do all the points  $\mathbf{P}_j$  belonging to the critical configuration. By intersecting these quadrics for two different values of  $k$ , we see that the centre of  $\mathbf{P}^i$  must lie on the intersection of two quadrics also passing through the points. It follows that if all points  $\mathbf{P}_j$  lie on a unique quartic, then all the camera centres must lie on this same quartic. Therefore, we can state:

- If  $\{\mathbf{P}^i, \mathbf{P}_j\}$  is a critical configuration with at least three cameras, then all the points  $\mathbf{P}_j$  must lie on the intersection of two ruled quadrics. If there is a unique such quadric intersection containing all the points, then it must contain all the camera centres as well.

One may wonder whether there are critical configurations

in which the quartic curve passing through the points is not unique, and in which all the points and cameras do not lie on a single elliptic quartic. The answer to this question is yes, according to ([9]). In that paper, it is shown that a configuration of six points and any number of cameras lying on a ruled quadric surface is critical. Note that six points are insufficient to define a unique quartic. In this case, the points and camera centres do not all lie on the same quartic curve.

## 11 Conclusion

We have described a major family of critical configurations for projective reconstruction from  $n \geq 2$  views. The main ambiguity is when all object points (regardless of how many) and all camera centres (again, regardless of the number of cameras) lie on an elliptic quartic curve. Furthermore, unlike the two-view case, there is a one-parameter family of conjugate reconstructions which give the same image projections.

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