Reconstruction of 3D-Curves from 2D-Images Using Affine Shape Methods for Curves*

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Abstract

In this paper, we propose an algorithm for doing reconstruction of general 3D-curves from a number of 2D-images taken by uncalibrated cameras. No point correspondences between the images are assumed. The curve and the view points are uniquely reconstructed, modulo common projective transformations and the point correspondence problem is solved. Furthermore, the algorithm is independent of the choice of coordinates, as it is based on orthogonal projections and aligning subspaces. The algorithm is based on an extension of affine shape of finite point configurations to more general objects.

1. Introduction

Affine shape of finite point configurations, has been introduced and studied in a series of papers by Sparr, see for example [8], where a reconstruction algorithm, for arbitrary numbers of points and images, was proposed. This was based on aligning subspaces by using orthogonal projections and maximising some of the largest eigenvalues of the sum of these projections. That algorithm is here extended and modified to handle curves as well. Furthermore, no point correspondences between the different images are needed. For simplicity, however, correspondences are assumed at the ends of each curve. Recent progress [2] has made it possible to relax this assumption. The algorithm is independent of the choice of coordinates and generic in numbers of images taken.

Another approach can be found in [5, 6] where a theory for computing the structure and motion of three-dimensional curves is developed. It is shown that the image of three-dimensional curves obey several constraints at each point. These constraints involve high order spatio-temporal derivatives of the image curve and camera motion. Thus, the derivatives of camera motion can, in theory, be found by using the derivatives of image curve motion. These derivatives are, however, difficult to obtain due to numerical problems.

Yet another contribution can be found in [7, 3, 1], where image-motion constraints that hold for certain points on the curve are developed. These constraints are more robust and can be applied to the silhouette of curved surfaces, but they do not exploit the full structure of the curve reconstruction problem.

The method developed here are somewhat similar to subspace methods developed for points [8] and as such they can be seen as a generalisation of [9] for points under orthography transformations to curves under perspective transformations.

2. Affine Shape

Our aim is to extend the algorithm for projective reconstruction of finite point configurations to reconstruction of general curves with no point correspondences known between the images, except at the ends of each curve. We would also like the conditions for projective reconstruction to be independent of the choice of coordinates, generic in the number of images, robust and easy to extend to recursive algorithms, where more and more images are taken into account. First we give a few definitions.

Let $\mathcal{X} = \{p_i\}_1^m \subset \mathbb{R}^n$ be an ordered point configuration in $\mathbb{R}^n$ and define the affine shape of $\mathcal{X}$ as the linear space

$$s(\mathcal{X}) = \left\{ \xi \;\middle|\; \sum_{i=1}^{m} \xi_i p_i = 0,\; \sum_{i=1}^{m} \xi_i = 0,\; \xi_i \in \mathbb{R} \right\}. \quad (1)$$

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It can be shown (see [8]) that $s(\mathcal{X})$ is a complete affine invariant, that is, there exists a nonsingular affine transformation a, such that $\mathcal{X}_1 = a(\mathcal{X}_2)$ if and only if $s(\mathcal{X}_1) = s(\mathcal{X}_2)$.

The definition (1) can be rewritten by introducing functions $f : I \to \mathbb{R}$ and a bijective map $\phi : I \to \mathcal{X} \subset \mathbb{R}^n$, which we call a **parametrisation** of $\mathcal{X}$. Then (1) becomes

$$s(\mathcal{X}) = \left\{ f \;\middle|\; \sum_{i \in I} f(i)\phi(i) = 0,\ \sum_{i \in I} f(i) = 0 \right\}$$

and for m points in $\mathbb{R}^n$, we could for example take $I = \{1, 2, \ldots, m\}$, $f(i) = \xi_i$ and $\phi(i) = p_i$. An extension to more general sets is done by substituting the sums with integrals.

Let $I \subseteq \mathbb{R}^k$ for some k be a compact set and denote the space of measurable functions $f : I \to \mathbb{R}$, such that

$$\|f\|_p = \left(\int_I |f|^p dx \right)^{1/p} < \infty,$$

by $L^p(I)$. Let $1/p + 1/q = 1$, with $1 \le p < \infty$ and let $f \in L^p(I)$ and $g \in L^q(I)$. Then the integral

$$\langle f | g \rangle_I = \int_I f(x) g(x) dx,$$

is defined and $L^p(I)$ and $L^q(I)$ are dual to each other. This means that, for every linear and continuous functional $T : L^p(I) \to \mathbb{R}$, there exists one and only one $u \in L^q(I)$, such that $T(\cdot) = \langle u | \cdot \rangle_I$ and vice versa.

Now let $\mathcal{X} \subseteq \mathbb{R}^n$, be a fairly arbitrary set as for example a connected curve or a surface in $\mathbb{R}^3$ and let $\phi : I \to \mathcal{X}$ be a parametrisation of $\mathcal{X}$. Then $\phi = (\phi_1, \ldots, \phi_n)$ and if $\phi_j \in L^q(I)$ for every j, we define the **affine shape** by

$$s(\phi) = \{ f \mid \langle f | \phi \rangle_I = 0,\ \langle f | 1 \rangle_I = 0,\ f \in L^p(I) \}.$$

This linear space of functions can of course be infinite dimensional. Moreover, the shape $s(\phi)$ is not entirely an intrinsic property of $\mathcal{X}$, as it changes with different parametrisations ϕ of $\mathcal{X}$. This is why we denote shape by $s(\phi)$ instead of $s(\mathcal{X})$ and this non intrinsic property causes problems when no point correspondences between the images are assumed.

We will now use $s(\phi)$ in the pinhole camera model. Let $\mathcal{Y}$ be the projective image of $\mathcal{X} \in \mathbb{R}^3$. Let $\psi : I \to \mathcal{Y}$ and $\phi : I \to \mathcal{X}$, be parametrisations such that $\psi(x)$ and $\phi(x)$ are corresponding points in the image and $\mathbb{R}^3$ respectively, for each $x \in I$. Then for some $\alpha : I \to \mathbb{R}$ and projection matrix P,

$$\alpha(x)\psi(x) = P\phi(x), \quad x \in I \tag{2}$$

and we call α the depth of ϕ. It can be shown that $\mathcal{Y}$ is the projective image of $\mathcal{X}$ with corresponding parametrisations ψ and ϕ respectively and depth α if and only if

$$\alpha s(\phi) \subseteq s(\psi). \tag{3}$$

This relation will be the starting point for a reconstruction algorithm. As noted above, the shapes of $\mathcal{X}$ and $\mathcal{Y}$ depend on the parametrisations ϕ and ψ. We refer to this as the **parametrisation problem** or **correspondence problem** and solving this for general curves in $\mathbb{R}^3$ is one of the contributions of this paper. For the moment assume that the point correspondences are known.

For this situation let $\mathcal{X}$ be a fixed object and $\{\mathcal{Y}_i\}_{i=0}^{m-1}$ a sequence of projective images of $\mathcal{X}$. Denote by $\phi : I \to \mathcal{X}$ and $\psi_i : I \to \mathcal{Y}_i$, parametrisations so that for some P_i and $\alpha_i : I \to \mathbb{R}$,

$$\alpha(x)\psi(x) = P\phi(x), \quad i = 0, 1, \ldots, m-1$$

holds for all $x \in I$. Then, by (2) and (3) for several images,

$$s(\phi) \subseteq \bigcap_{i=0}^{m-1} \frac{1}{\alpha_i} s(\psi_i),$$

or equivalently, by multiplying both sides by α_0,

$$\alpha_0 s(\phi) \subseteq s(\psi_0) \bigcap_{i=1}^{m-1} q_i s(\psi_i),$$

where $q_i = \alpha_0/\alpha_i$ are called **kinetic depths**. Since $s(\phi)$ is usually unknown, we replace $\alpha_0 s(\phi)$ by $s(\phi)$ and get the relation

$$s(\phi) \subseteq s(\psi_0) \bigcap_{i=1}^{m-1} q_i s(\psi_i). \tag{4}$$

This freedom in replacing $\alpha_0 s(\phi)$ by $s(\phi)$ is just the *non uniqueness* of projective reconstructions. As reconstructions, any element in the orthogonal complement of $s(\phi)$ will do and all other equally valid reconstructions can be obtained by multiplying both sides of (4) with functions α, such that the product is still a shape space, that is, all its elements have integral zero.

3. Applications to curve reconstruction

Assume that $\mathcal{X} \subset \mathbb{R}^3$ is a curve, with finite extension, such that $\mathcal{X}$ does not belong to any affine plane and assume that the images $\mathcal{Y}_i$ are taken from general locations and not only from some affine plane. Let $p = 2$ and $I = [0, 1]$. Then $L^2(I)$ is a **Hilbert** space and because of the generality of the curve and camera positions, a dimensionality argument (or rather by co-dimensionality) implies that (4) becomes the equality

$$s(\phi) = s(\psi_0) \bigcap_{i=1}^{m-1} q_i s(\psi_i), \tag{5}$$

or by considering the orthogonal complement

$$s^{\perp}(\phi) = s^{\perp}(\psi_0) + \sum_{i=1}^{m-1} q_i^{-1} s^{\perp}(\psi_i), \tag{6}$$

where $q_i^{-1} = 1/q_i$ and the sum of two linear subspaces is defined by $A+B = \{a+b \mid a \in B,\ b \in B\}$. In fact, since $\mathcal{X}$ is a general curve, $s(\phi)$ has codimension equal to four and in the same way $q_i s(\psi_i)$ have codimensions three for all i and since these $q_i s(\psi_i)$ do not coincide the right hand side of (4) has codimension four as well.

Assume that the curves are open, that is they have different starting and ending points. This condition is needed since we have to be able to make correspondences between these end points in different images, but this is also all we need. The points in between on the curves are now not assumed to have known correspondences. The problem is then to find both parametrisations of $\mathcal{Y}_i$ (i.e ψ_i) and the kinetic depths q_i in (6) for obtaining a reconstruction $s^{\perp}(\phi)$.

Introduce the following notation. Let P_X be the orthogonal projection onto $s(\phi)$, P_i the orthogonal projection onto $q_i s(\psi_i)$ and let Q_X be the orthogonal projection onto $s^{\perp}(\phi)$, and similarly Q_i the orthogonal projection onto $q_i^{-1} s^{\perp}(\psi_i)$. These projection operators can be explicitly written using orthonormal bases. For example, let $\mathcal{X}$ be written in extended coordinates, that is, we add the number 1 as a fourth coordinate for each point and let $\phi = (\phi_1, \phi_2, \phi_3, \phi_4) : I \to \mathcal{X}$ be a parametrisation with $\phi_4(t) \equiv 1$. Let $\{\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3, \tilde{\phi}_4\}$ be an orthonormal basis for $\mathrm{lh}(\phi_1, \phi_2, \phi_3, \phi_4)$, where lh denotes the *linear hull* or *linear span* of a set. Then the projection operators can explicitly be written as

$$Q_X(f) = \sum_{k=1}^{4} \langle \tilde{\phi}_k | f \rangle \tilde{\phi}_k$$

and as $P_X(f) = (I - Q_X)(f)$. The condition (5) and (6) can be rewritten using the projection operators as either of the following four conditions

- The operator $\frac{1}{m} \sum_{i=0}^{m-1} P_i$ acts as the unity operator on $s(\phi)$.
- The operator $P_X \frac{1}{m} \sum_{i=0}^{m-1} P_i$ is equal to P_X
- The operator $\frac{1}{m} \sum_{i=0}^{m-1} Q_i$ has only four non-zero eigenvalues.
- The operator $P_X Q_i$ is the zero operator for every i.

These equalities will never hold exactly due to noise and other errors. It is of interest to introduce error criterias to minimise. One interesting application of the theory of affine shape is the following observation.

Any criteria that is based on the linear spaces or the projection operators above is *invariant* of the choice of affine coordinate system in the images. Any such criteria also has the property that all images are treated in a symmetrical fashion and is generic in the number of images taken. Such an invariant criteria or measure, will be called a **proximity measure**.

There are several possibilities. One possibility is the following proximity measure, based on the fact that $P_X Q_i = 0$ for all i,

$$\mu = \sum_{i=0}^{m-1} \|P_X Q_i\|_{HS}^2 \ ,$$

where the the operator norm HS is the **Hilbert-Schmidt norm**, see [4]. For finite dimensional spaces, this norm is the same as the **Frobenius** norm. The HS-operator norm is given by

$$\|A\|_{HS}^2 = \sum_k \|Ae_k\|^2 \ ,$$

where e_i is an orthonormal basis for $L^2(I)$. The norm is independent of the choice of basis. By choosing this basis such that the three first components $\{e_1, e_2, e_3\}$ span $q_i^{-1} s^{\perp}(\psi_i)$ (and consequently $P_X Q_i e_k = 0$ for all $k > 3$), it is seen that

$$\|P_X Q_i\|_{HS}^2 = \sum_{k=1}^{3} \|P_X e_k\|^2 \ .$$

Thus, if $\{\tilde{\psi}_{i,1}, \tilde{\psi}_{i,2}, \tilde{\psi}_{i,3}\}$ is an orthonormal basis of each $q_i^{-1} s^{\perp}(\psi_i)$ it follows that

$$\mu = \sum_{i=0}^{m-1} \sum_{k=1}^{3} \|P_X \tilde{\psi}_{i,k}\|^2 \ .$$

4. Algorithm construction

In the following, let $I = [0,1]$ and let all the parametrisations ϕ and ψ_i be in extended coordinates and we set $Y_i = \mathrm{lh}\{\psi_{i1} q_i^{-1}, \psi_{i2} q_i^{-1}, \psi_{i3} q_i^{-1}\}$ and $X = \mathrm{lh}\{\phi_1, \phi_2, \phi_3, \phi_4\}$. These spaces are usually referred to as depth spaces, [8].

We propose the following algorithm which is based on repeatedly finding $s(\phi)$, adjusting the kinetic depths q_i and the parametrisations of $\mathcal{Y}_i$.

I **Initialise**: Choose one parametrisation ψ_i in each image curve by for example using the image based arc-length. Set $q_i(x) = 1$ and $Y_i = \mathrm{lh}(\psi_i)$, for all i.

II **Update** X: Keeping Y_i fixed for all i, find P_X that minimises μ.

III **Update** q_i: Keeping X and Y_i fixed, find q_i such that $q_i^{-1} Y_i$ minimises μ. Set $Y_i := q_i^{-1} Y_i$.

IV **Update parametrisation**: Keeping X and Y_i fixed, find a continuous bijection $\gamma_i : I \to I$, such that $Y_i \circ \gamma_i$ minimises μ. Set $Y_i := Y_i \circ \gamma_i$ and go to II.

It is difficult to minimise μ with respect to all parameters. It is, however, reasonably fast to solve each of the three steps II to IV approximately as will be demonstrated. Furthermore, since we iterate the procedure over and over, we do not have to be very precise in each step.

4.1. Step I. Initialisation

Let each image curve $\psi_i(t) = (\psi_{i1}(t), \psi_{i2}(t), 1)$ be parametrised using scaled image arclength $t \in I$ so that $\psi_i(0)$ and $\psi_i(1)$ are the endpoints and such that $(\psi_{i1}')^2 + (\psi_{i2}')^2$ is constant. Let the depth initially be $q_i(t) = 1$ for all points in all curves and $Y_i = \mathrm{lh}(\psi_{i,1}, \psi_{i,2}, 1)$. Furthermore, we use the proximity measure μ, when describing each step of the algorithm in more detail below.

4.2. Step II. Computation of X given Y_i

Let $\{\psi_{i,1}, \psi_{i,2}, \psi_{i,3}\}$ be an orthonormal basis for the 3-dimensional linear space Y_i. The 4-dimensional linear space $X = s^{\perp}(\phi)$ corresponding to the reconstructed curve is then given by the linear span of all basis functions $\psi_{i,k},\ i = 0, \dots, m-1,\ k = 1, 2, 3$.

$$X = \mathrm{lh}\{\psi_{i,k},\ i = 0, \dots, m-1,\ k = 1, 2, 3.\}$$

Even if the basis functions $\psi_{i,k}$ are slightly incorrect due to measurement errors, slightly incorrect kinetic depths or incorrect parametrisations, an approximate basis for X can be found by solving

$$\min_{\dim X = 4} \mu = \min_{\dim X = 4} \sum_{i=0}^{m-1} \sum_{k=1}^{3} \| P_X \psi_{i,k} \|^2 \ . \qquad (7)$$

This can be solved using singular value decomposition. Form the symmetric matrix

$$M_1 = \begin{pmatrix} \langle \psi_{0,1} | \psi_{0,1} \rangle & \dots & \langle \psi_{0,1} | \psi_{m-1,3} \rangle \\ \langle \psi_{0,2} | \psi_{0,1} \rangle & \dots & \langle \psi_{0,2} | \psi_{m-1,3} \rangle \\ \langle \psi_{0,3} | \psi_{0,1} \rangle & \dots & \langle \psi_{0,3} | \psi_{m-1,3} \rangle \\ \langle \psi_{1,1} | \psi_{0,1} \rangle & \dots & \langle \psi_{1,1} | \psi_{m-1,3} \rangle \\ \vdots & \ddots & \vdots \\ \langle \psi_{m-1,3} | \psi_{0,1} \rangle & \dots & \langle \psi_{m-1,3} | \psi_{m-1,3} \rangle \end{pmatrix} .$$

Compute a singular valued decomposition $M_1 = USV^T$, where U and V are orthogonal matrices and S is a non-negative diagonal matrix. The matrix M_1 should have rank 4, but if it is not, then the matrix $\hat{M}$, which is closest in Frobenius norm with rank 4, is obtained as $\hat{M} = US_4V^T$, where S_4 is obtained from S by setting all but the four largest diagonal elements in S to zero. An orthonormal basis for X can then be given as

$$\phi_k = \frac{1}{\sqrt{S_{k,k}}} (V_{1,k}\psi_{0,1} + V_{2,k}\psi_{0,2} + V_{3,k}\psi_{0,3} + V_{4,k}\psi_{1,1} + \dots + V_{3m,k}\psi_{m-1,3}), \qquad k = 1, 2, 3, 4 \ . \quad (8)$$

This X solves the optimisation problem (7).

4.3. Step III. Computation of kinetic depths

Let $\{\psi_{i1}, \psi_{i2}, \psi_{i3}\}$ be an orthonormal basis for Y_i and let $\{\phi_1, \phi_2, \phi_3, \phi_4\}$ be an orthonormal basis for the 4-dimensional linear space $X = s^{\perp}(\phi)$, corresponding to the reconstructed curve. Introduce the projection operator P_X orthogonal to X according to

$$P_X f = f - \sum_{j=1}^{4} \phi_j \langle \phi_j | f \rangle .$$

We want to find q_i^{-1} so that $P_X \psi_{ik} q_i^{-1}$ is small, e.g. by minimising

$$\sum_{k=1}^{3} \left\| P_X \psi_{ik} q_i^{-1} \right\|^2 ,$$

for all q_i^{-1} with $\|q_i^{-1}\| = 1$. For convenience we drop the index i for the index of the image, since each image is treated separately. Parametrise q^{-1} using a finite basis f_j according to

$$\mathbb{R}^n \ni x \to q^{-1}(x) = \sum_{j=1}^{n} x_j f_j \ .$$

The projection $P_X \psi_k q^{-1}(x) = \sum (P_X(\psi_k f_j)) x_j = \sum_j \vartheta_{k,j} x_j$ should be close to zero, where we have introduced $\vartheta_{kj} = P_X(\psi_k f_j)$.

The least squares solution with respect to the scalar product can be found by singular value decomposition USV^T of the following matrix

$$M_2 = \sum_{k=1}^{3} \begin{pmatrix} \langle \vartheta_{k,1} | \vartheta_{k,1} \rangle & \dots & \langle \vartheta_{k,1} | \vartheta_{k,N} \rangle \\ \vdots & \ddots & \vdots \\ \langle \vartheta_{k,N} | \vartheta_{k,1} \rangle & \dots & \langle \vartheta_{k,N} | \vartheta_{k,N} \rangle \end{pmatrix} .$$

By taking x as the last column of V we obtain the vector x of unit length which minimises

$$\min_{\|x\|=1} \sum_{k=1}^{3} \left\| P_X \psi_{ik} q_i^{-1}(x_i) \right\|^2 .$$

Set $Y_i = \mathrm{lh}\{q_i^{-1}\psi_{i1}, q_i^{-1}\psi_{i2}, q_i^{-1}\psi_{i3}\}$.

4.4. Step IV. Reparametrise each image curve

Let $\{\psi_{i1}, \psi_{i2}, \psi_{i3}\}$ be an orthonormal basis for Y_i and $\{\phi_1, \phi_2, \phi_3, \phi_4\}$ an orthonormal basis for the 4-dimensional linear space $X = s^{\perp}(\phi)$, corresponding to the reconstructed curve. We want to find a reparametrisation γ_i in each image i, such that $P_X(\psi_{ik} \circ \gamma_i)$ is small by minimising

$$\sum_{k=1}^{3} \|P_X(\psi_{ik} \circ \gamma)\|^2$$

over some set of reparametrisations. Again, we drop the index i for convenience, since each image i can be treated independently. Parametrise γ using a finite basis g_j according to

$$\mathbb{R}^n \ni x \to \gamma(t, x) = t + \sum_j x_j g_j(t)$$

where the basis function fulfill $g_j(0) = 0$ and $g_j(1) = 0$. The function $\gamma(t, x)$ is monotonic for small x such that

$$\frac{\partial \gamma}{\partial t} = 1 + \sum_j x_j g_j'(t) > 0, \quad 0 \le t \le 1.$$

A conservative bound on $|x|$ is to take

$$|x|^2 < \frac{1}{\max_t \sum_j |g_j(t)|^2} . \tag{9}$$

Now study the linearisation of $\Theta_k(x) = P_X(\psi_k \circ \gamma(x))$ around $x = 0$,

$$\Theta_k(x) \approx \Theta_k(0) + \nabla_x \Theta_k(0)\ x .$$

The derivatives are given by

$$\theta_{k,j} = \left. \frac{\partial \Theta_k}{\partial x_j} \right|_{x=0} = P_X(\psi_k' g_j) .$$

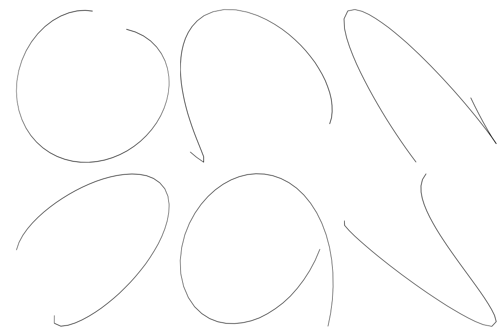

Figure 1: Six images of a curve.

The *Gauss-Newton* iteration for the minimisation problem

$$F(x) = \min_x \sum_{k=1}^{3} \|\Theta_k(x)\|^2$$

is given by the *normal equations* $Ax = b$, where

$$A = \sum_{k=1}^{3} \begin{pmatrix} \langle \theta_{k,1} | \theta_{k,1} \rangle & \dots & \langle \theta_{k,1} | \theta_{k,N} \rangle \\ \vdots & \ddots & \vdots \\ \langle \theta_{k,N} | \theta_{k,1} \rangle & \dots & \langle \theta_{k,N} | \theta_{k,N} \rangle \end{pmatrix}$$

and

$$b = \sum_{k=1}^{3} \begin{pmatrix} \langle \theta_{k,1} | \Theta_{ik} \rangle \\ \vdots \\ \langle \theta_{k,N} | \Theta_{ik} \rangle \end{pmatrix} .$$

If the solution of the normal equations gives an x that is to large according to (9) or if $F(x)$ is larger than $F(0)$ due to the non-linearities of the function F, then it is always possible to decrease the error function by restricting the step length since A is positive definite and therefore x is a descent direction.

The result is a reparametrisation of the basis for Y_i and we set

$$Y_i = \mathrm{lh}\{\psi_{i1} \circ \gamma_i(x), \psi_{i2} \circ \gamma_i(x), \psi_{i3} \circ \gamma_i(x)\} .$$

5. Experimental Validation

The following experimental validation was performed. A simulation was made with 6 images of a curve as shown in Figure 1.

In each image i the curve was parametrised according to arclength $(\psi_{i1}(s), \psi_{i2}(s), 1)$. Note that the start and endpoints of these curves were assumed to be known, but that points with the same curve parameter s in different images are not necessarily in correspondence. For each curve $\mathcal{Y}_i$ a space $Y_i = \mathrm{lh}\{\psi_{i,1}, \psi_{i,2}, 1\}$

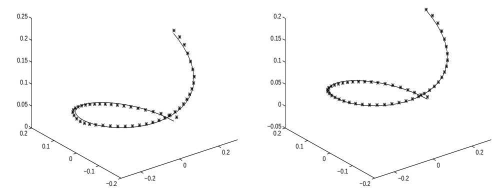

Figure 2: The reconstructed curve (*) and ground truth (-) after the first and the 20'th iteration.

was chosen. Twenty iterations of the algorithm were then performed and after each step, the proximity measure μ was stored, and the reconstructed curve was compared with ground truth. Figure 2 shows the reconstructed curve after the first and after the 20'th iteration. Notice the relatively good alignment already after the first iteration.

Since $L^2(I)$ is infinite in dimension, it is convenient to consider a subspace of finite dimension and approximate the "real" functions, with some function in U. This makes it easier to represent functions by expansions in a basis of U. In the experiments, two different choices of finite subspaces U of $L^2(I)$ were tried. One choice is to use finite sampling in the frequency domain, i.e. each component is represented by Fourier coefficients up to a certain order. Another choice of representation is to use finite sampling in the spatial domain. Each function $f \in L^2(I)$ is thus approximated with a piecewise constant function.

6. Conclusions

In this paper we have presented an extension of affine shape to measurable sets and shown its applicability to the reconstruction of three-dimensional curves from its two-dimensional projections. The resulting algorithms seem to give reasonable good estimates already after a few iterations. The convergence properties of the algorithms have, however, not been examined thoroughly. This is an interesting aspect for continued research.

The idea of using proximity measures that are invariant under choice of coordinate systems is very appealing. However, if additional information are given, e.g. about the noise of the measured image curves, then this could be used to estimate structure and motion on a statistical basis. Such algorithms typically involve optimisation and can be sensitive on the choice of initial estimates. The algorithm presented in this paper seems to be efficient at giving good initial solutions. An extension to incorporate such statistical refinements will be presented in [2].

Another interesting and important aspect is to incorporate further experiments on real image sequences to obtain more information about the strengths and weak points of the algorithm. Although the algorithms used so far assume that the two end-points of the curve can be detected in each image, it is possible to extend the theory to deal with closed curves and partially occluded curves. This investigation, [2], will be continued.

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