Fundamental Difficulties with Projective Normalization of Planar Curves ⋆

Kalle Åström

Dept. of Mathematics, Lund Institute of Technology, Box 118, S-221 00 Lund, Sweden

Abstract. In this paper projective normalization and projective invariants of planar curves are discussed. It is shown that there exists continuous affine invariants. It is shown that many curves can be projected arbitrarily close to a circle in a strengthened Hausdorff metric. This does not infer any limitations on projective invariants, but it is clear that projective normalization by maximizing compactness is unsuitable. It is also shown that arbitrarily close to each of a finite number of closed planar curves there is one member of a set of projectively equivalent curves. Thus there can not exist continuous projective invariants, and a projective normalisation scheme can not have both the properties of continuity and uniqueness. Although uniqueness might be preferred it is not essential for recognition. This is illustrated with an example of a projective normalization scheme for non-algebraic, both convex and non-convex, curves.

1 Model Based Vision Using Invariants

The pinhole camera is often an adequate model for projecting points in three dimensions onto a plane. Using this model it is straightforward to predict the image of a collection of objects in specified positions. The inverse problems, to identify and to determine the three-dimensional positions of possible objects from an image, are however much more difficult. Traditionally recognition has been done by matching each model in a model data base with parts of the image. Recently, model based recognition using viewpoint invariant features of planar curves and point configurations has attracted much attention, cf. [MZ1]. Invariant features are computed directly from the image and used as indices in a model data base. This gives algorithms which are significantly faster than the traditional methods. These techniques cannot, however, be used to recognise general curves or point features in three dimensions by means of one single image. Additional information, e.g. that the object is planar, is needed. For point configurations the reason is that only trivial invariants exist in the general case, as is shown in [BW1]. In this paper it is shown that there are some fundamental limitations also for planar curves.

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The paper is organized as follows. In Section 2 the notation is introduced. Normalization schemes are discussed in a general framework and their relation to invariants, recognition and pose determination are given. A classical normalization scheme under affine transformations based on moments is presented in Section 3. The key observation is that this normalization scheme is continuous in the Hausdorff metric. In Section 4 it is shown that every curve in a large class can be projectively transformed into a curve arbitrarily close to a circle in a strengthened Hausdorff metric. A direct consequence is that projective normalization by maximizing compactness is inherently difficult and the normalized curve will depend crucially on how the boundary curve is represented. A somewhat surprising fact is shown in Section 5. Given a finite number of closed planar curves $\Gamma_1, \ldots, \Gamma_m$ it is possible to construct another set of planar curves $D_1, \ldots, D_m$ which are projectively equivalent and in which D_i in the Hausdorff metric is arbitrarily close to Γ_i, $i = 1, \ldots, m$. One consequence is that there exists no non-constant continuous projective invariants from the set of planar curves to the real line. Another consequence is that normalization schemes on closed planar curves can not both be continuous in this metric and give a unique representative from each equivalence class. In Section 6 a normalization scheme for non-algebraic, both convex and non-convex curves is presented and illustrated. This scheme is by no means perfect, but it illustrates that uniqueness can be sacrificed for continuity.

2 Preliminaries

This section contains some preliminaries and notations. First the problem of extracting geometrical features from an image is briefly discussed. The main idea is that it is possible to find small regions, despite occlusion and changes in lighting. The idea of using normalization to find invariants is then described and some notations are introduced.

2.1 Extraction of Curves

A grayscale image contains large amounts of information. The main idea of invariant based recognition is to throw away information that varies with lighting, occlusion and viewpoint, and to keep invariant features that allow recognition. The first step in this process is to extract geometrical features in the image. This can be done by algorithms for edge extraction and segmentation, cf. Figure 1. In this figure concavities are extracted from the outline of a spanner. In this paper we discuss the possibility of finding stable viewpoint invariant features of regions or closed curves.

2.2 Using Normalization to Find Invariants

A group G, in this paper the planar affine or projective group, is said to act on a set Ω if there exists a mapping $(G, \Omega) \ni (g, \omega) \longrightarrow g(\omega) \in \Omega$ with properties

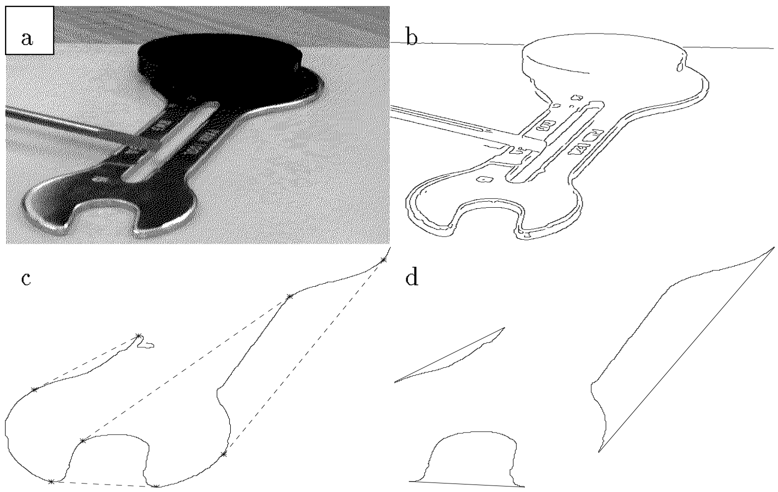

Fig. 1. 1a: A grayscale image of a scene with a roughly planar object. 1b: Edges are extracted using a Canny-Deriche edge detector. 1c: Distinguished points on one edge are used to segment a curve into pieces in a projectively invariant way. 1d: Distinguished points and lines can also be used to extract small regions in a projectively invariant way. Three such regions are shown in the figure.

$1(\omega) = \omega$, $\forall \omega \in \Omega$ and $g_1(g_2(\omega)) = (g_1 g_2)(\omega)$, $\forall \omega \in \Omega$, $\forall g_1, g_2 \in G$. The notation for group action is either $g\omega$ or $g(\omega)$.

Two elements ω_1 and ω_2 are said to have the same shape if $\omega_1 = g\omega_2$ for some transformation $g \in G$. This is an equivalence relation, because of the group structure of G. We write

$$\omega_1 \sim \omega_2 \iff \exists g \in G, \quad \omega_1 = g\omega_2 \ . \tag{1}$$

The equivalence relation divides Ω into disjoint equivalence classes. Denote the equivalence class containing ω by $G\omega = \{g\omega | g \in G\}$.

Let $T : \Omega \longrightarrow W$ be a function defined on Ω with values in some feature set W. This function is called an *invariant* if $\omega_1 \sim \omega_2 \Longrightarrow T(\omega_1) = T(\omega_2)$ and a *complete invariant* if $\omega_1 \sim \omega_2 \iff T(\omega_1) = T(\omega_2)$.

A *normalization scheme* is simply a choice of *normal* reference frames. Let Ω_P denote this set of normal elements. One common construction is $\Omega_P = \{\omega | P(\omega) = 0\}$, where $P : \Omega \rightarrow R^n$ is some function. Another construction is to let Ω_P be those elements which maximize some feature in its equivalence class. For each element ω let the corresponding equivalence class $G\omega$ be represented by its normal elements, i.e. by

$$T(\omega) = G\omega \cap \Omega_P \ . \tag{2}$$

In the sequel we will say that a normalization scheme has the *uniqueness property* if there is only one normal reference frame, i.e. $G\omega \cap \Omega_P$ has only one element. A normalization scheme is called *continuous* if the normal reference frames depend continuously on ω. Assume that we have uniqueness in the normalization scheme. Any element ω_1 can then be uniquely factorized as

$$\omega_1 = g_1 \omega_1^{inv}, \tag{3}$$

with $g_1 \in G$ and $\omega_1^{inv} = T(\omega_1)$.

Isotropy, cf. [Wi1, Gå1], and maximal compactness, cf. [BY1] are two examples of affine normalization of planar curves. These two ideas give the same normal reference frames. This reference frame is unique up to similarity transformations.

2.3 Notations

A *rectifiable* curve is a continuous parametric curve with finite arclength. Let $\mathcal{C}$ be the set of all closed rectifiable curves. For such curves it is possible to calculate the arclength l and the area A enclosed by the curve. It is a well known fact from the calculus of variations that $l(C)^2/A(C) \geq 4\pi$, with equality if and only if C is a circle. For a specific curve $C \in \mathcal{C}$, let P_C be the set of projective transformations that sends C into $\mathcal{C}$. In other words such transformations do not send any of the points of C to infinity. Two images of the same planar curve, caught by a pinhole camera, are related by such a transformation. Two metrics on $\mathcal{C}$ are defined by

$$d(C_1, C_2) = \max_{z_1 \in C_1} \min_{z_2 \in C_2} ||z_1 - z_2|| + \max_{z_2 \in C_2} \min_{z_1 \in C_1} ||z_1 - z_2|| \tag{4}$$

and

$$\tilde{d}(C_1, C_2) = \max_{z_1 \in C_1} \min_{z_2 \in C_2} ||z_1 - z_2|| + \max_{z_2 \in C_2} \min_{z_1 \in C_1} ||z_1 - z_2|| + |l(C_1) - l(C_2)|. \tag{5}$$

Here $||x||$ is the euclidean norm. The first metric is the ordinary Hausdorff metric. The second one is a strengthened version, the modification being that also the arclengths should be compared. These metrics will be used to compare two projected curves in the image plane. Due to digitization effects and other errors in the image plane, it is difficult to discriminate two image curves that are close in this metric. Every point on each curve is close to some point on the other curve and the arclengths are almost equal.

Let Ω be the class of all compact sets $\omega \subset R^2$ with positive area, whose boundary $C = \partial\omega$ is in $\mathcal{C}$. Two elements of Ω are compared using the above metrics on the boundary, i.e. we define $d(\omega_1, \omega_2) = d(\partial\omega_1, \partial\omega_2)$. Let moments be defined as

$$\begin{aligned} m_0(\omega) &= \int_{x\in\omega} dx_1 dx_2 \\ m_1(\omega)_i &= \int_{x\in\omega} x_i \, dx_1 dx_2 \\ m_2(\omega)_{ij} &= \int_{x\in\omega} x_i x_j \, dx_1 dx_2 \\ m_3(\omega)_{ijk} &= \int_{x\in\omega} x_i x_j x_k \, dx_1 dx_2 \end{aligned}$$

Notice that m_0 is a scalar, m_1 a vector and m_2 a matrix. The moments depend continuously on ω in the metrics above.

The planar affine transformation group G_a and the planar projective transformation group G_p are used. For simplicity we will talk somewhat losely about G_p acting on Ω.

Once in a normal reference frame any feature is invariant. Moments and Fourier coefficients can be used for curves and regions. In our experiments we have divided the plane in sectors and used the area of a region in each sector as a feature, see [Ås2, Ås3]. This has been quite effective.

Assume that d is a metric on Ω. Assume also that we have a normalization scheme with uniqueness. Then a G-invariant metric on the class of shapes is defined by

$$d_G(\omega_1, \omega_2) = d(T(\omega_1), T(\omega_2)) \tag{6}$$

Rotationally symmetric shapes are difficult to normalize with respect to rotations. Any rotational normalization scheme will have trouble with shapes that are close to rotational symmetry. On the other hand rotations do not affect Euclidean distance between a pair of points. It is therefore possible to define a metric on shapes with respect to the group G_{rot} of rotational transformations

$$G_{rot} = \{g = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} | \theta \in R\}$$

acting on R^2 by left multiplication. The following induced Hausdorff metric will do

$$d_{rot}(\omega_1, \omega_2) = \min_{g \in G_{rot}} d(g\omega_1, \omega_2) \tag{7}$$

3 Existence of Continuous Affine Invariants

The following affine normalization scheme is based on the well known principle of moments of inertia. The key issue here is that the normalization scheme is continuous in the Hausdorff metric. The results below generate corresponding results for the closed boundary curves of such regions. A typical example is an extracted concavity as in Figure 1.

The moments change in a simple way when a region is transformed affinely. For instance we have

$$\begin{array}{rcccl} m_1(\omega + b) = & \int_{y \in \omega + b} y \, dy & = & \int_{x \in \omega} (x + b) \, dx & = m_1(\omega) + b m_0(\omega) \\ m_1(A\omega) = & \int_{y \in A\omega} y \, dy & = & \int_{x \in \omega} Ax |\det A| \, dx & = |\det(A)| A m_1(\omega) \\ m_2(A\omega) = & \int_{y \in A\omega} y y^T \, dy & = & \int_{x \in \omega} Ax (Ax)^T |\det A| \, dx & = A m_2(\omega) A^T |\det(A)| \end{array}$$

It is therefor easy to use the moments to select representatives from each equivalence class.

Theorem 1. *Given $\omega \in \Omega$, there is an orientation preserving affine transformation $x \mapsto Ax + b$ unique up to rotations that transforms ω into a region $\omega' \in \Omega_P$, where*

$$\Omega_P = \{\omega | \quad m_0(\omega) = 1, \quad m_1(\omega) = 0, \quad m_2(\omega) = aI, \quad a \in R\}. \tag{8}$$

Furthermore the complete invariant $T(\omega) = G_{aff}\omega \cap \Omega_P$ is a continuous mapping from (Ω, d) to $(\Omega/G_{rot}, d_{rot})$.

Proof. First translate so that the center of mass is at the origin. The condition $m_1(\omega + b) = 0$ gives

$$m_1(\omega + b) = m_1(\omega) + bm_0(\omega) = 0.$$

Since $m_0(\omega) \neq 0$, b is uniquely determined as

$$b = -\frac{m_1(\omega)}{m_0(\omega)}.$$

Assuming that $m_1(\omega) = 0$, A has to be chosen so that the second moment is the identity matrix. Observe that $m_1(\omega) = 0$ implies that $m_1(A\omega) = 0$, i.e. the mass center is not affected by multiplication with A. The condition

$$m_2(A\omega) = |\det(A)|Am_2(\omega)A^T = aI$$

gives

$$m_2(\omega) = BB^T a|\det(B)|$$

with $B = A^{-1}$. Let $|B|$ be the positive square root of the positive definite matrix BB^T, i.e. $|B|^2 = BB^T$. Then $|B|$ is a scalar multiple of $\sqrt{m_2(\omega)}$. The matrix A is thus given as

$$A = |B|^{-1}.$$

It is determined uniquely up to rotation and scale. Finally fix the scale by $m_0(\omega') = 1$. It is easy to see that all transformations are continuous in the Hausdorff metric.

The normalization scheme based on (8) has the following properties.

- Uniqueness. In (3), g and ω^{inv} are unique up to rotation.
- Continuity. Both g and ω^{inv} depend continuously on ω.
- Easy to compute. As can be seen from the proof of Theorem 1, the transformation g can be directly computed from the moments of order 0, 1 and 2 of the region ω.
- Robust to digitization errors.
- No distinguished points are needed.

This scheme can be very useful in the recognition of planar curve segments, obtained from concavities. The affine approximation is often valid since these concavities often occupy a small region in the image. It can be seen that the method has good robustness properties, in comparison with maximum compactness and weak isotropy.

4 Inherent Difficulties in Maximizing Compactness

Let $\widetilde{\mathcal{C}} \subset \mathcal{C}$ consist of those curves in $\widetilde{\mathcal{C}}$ such that the boundary of the convex hull has at least one smooth, curved part. In this section it will be proved that all curves $\mathcal{C}$ can be projectively transformed into a curve arbitrarily close to a circle even in the strengthened Hausdorff metric $\tilde{d}$. This fact does not by itself have any implications on the existence of projective invariants.

Theorem 2. *Let C_0 be a circle of radius one. Then*

$$C \in \widetilde{\mathcal{C}} \Longrightarrow \inf_{p \in P_C} \tilde{d}(p(C), C_0) = 0.$$

One interpretation of this theorem is that comparing the images of C from some sequence of projective viewpoints, these images look more and more like a circle. As will be seen in the proof below the projective transformations involved when approaching the limit are quite extreme, but still non-singular.

Proof. Choose a point $a \in C$ so that C is smooth at a, and so that the tangent at a intersects C only at a. Choose a coordinate system with origin at a, with x-axis along the tangent, and so that the curvature at a equals one.

The idea of the proof is to construct a sequence of transformations $(p_n)_1^\infty$ so that $p_n(C) \longrightarrow C_0$ as $n \longrightarrow \infty$, in the metric $\tilde{d}$. The image of a part of the curve around a will form the main part of C_0, and the remaining part of C will be mapped into a neighbourhood of one particular point of C_0.

The transformations $p_n, n > 0$ are defined by

$$p_n(x, y) = \left(\frac{2nx}{(n^2-1)y+2}, \frac{2n^2y}{(n^2-1)y+2} \right). \tag{9}$$

We will also use the ellipses

$$C_\epsilon = \{((1+\epsilon)\cos t, \sin t + 1) | t \in \mathcal{R}\}, \quad \epsilon > -1 \tag{10}$$

with center at the point $(0, 1)$, axis of length $1 + \epsilon$ in the x-direction and of length 1 in the y-direction. In particular, C_0 is the unit circle $x^2 + (y-1)^2 = 1$. These ellipses intersect twice at $(0, 0)$ and twice at $(0, 2)$.

One can easily verify, e.g. using homogeneous coordinates, that the family $(p_n)_1^\infty$ has the following properties:

$$p_1 = \text{identity} \tag{11}$$

$$p_a \circ p_b = p_{ab} \tag{12}$$

$$p_n(C_\epsilon) = C_\epsilon \tag{13}$$

$$p_n(0,0) = (0,0) \tag{14}$$

$$p_n(0,2) = (0,2) \tag{15}$$

By (13), the transformations p_n reparametrise the ellipses C_ϵ. It will be seen that if $n > 1$ a vicinity around $(0,0)$ expands and a vicinity around $(0,2)$ contracts. More precisely, by rewriting (9) as

$$p_n(x,y) = \left(\frac{2nx}{(n^2-1)y+2}, 2 + \frac{2y-4}{(n^2-1)y+2} \right), \tag{16}$$

it follows that for every compact region D in the open half plane $\{(x,y) |\ y > 0\}$,

$$\sup_D |p_n(x,y) - (0,2)| \le K/n, \tag{17}$$

for some constant K. Hence $(p_n)_1^\infty$ is uniformly convergent to $(0,2)$ on D. Since the jacobians of p_n are uniformly bounded by $O(1/n)$ on D, it also follows that the transformations p_n are uniformly Lipschitz continuous on D, i.e.

$$|p_n(x_1,y_1) - p_n(x_2,y_2)| \le K/n|(x_1,y_1) - (x_2,y_2)|, \\ \forall (x_1,y_1) \in D, \forall (x_2,y_2) \in D, \forall n \tag{18}$$

The inverse projective transformation p_n^{-1} is equal to p_n after the change of variables $x \mapsto -x, y \mapsto 2-y$, which exchanges the points $(0,0)$ and $(0,2)$. Thus the inverse transformations p_n^{-1} also have the contractive properties (17) and (18) in every compact region D in the open half plane $\{(x,y) |\quad y < 2\}$.

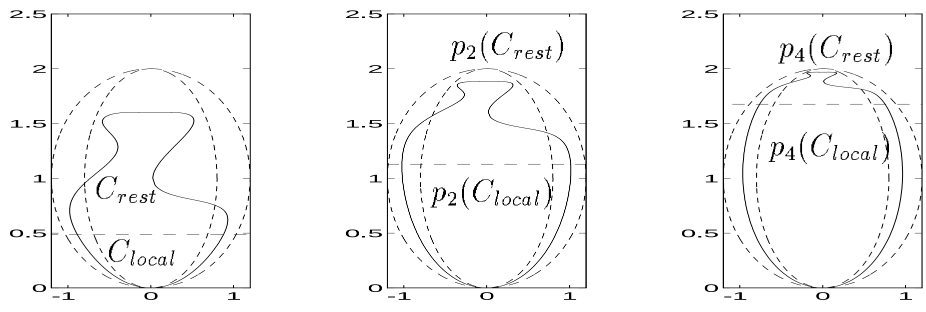

Fig. 2. The curve is split into two parts. A local part C_{local} belongs to the region bounded by the line and the two ellipses. C_{rest} is the complementary part of C.

Take $\epsilon > 0$, and let C_{local} be the connected component of C in a neighbourhood of $(0,0)$, that lies between the ellipses C_ϵ and $C_{-\epsilon}$, cf. Figure 2. Since the curve $p_n(C_{local})$ lies between the ellipses, the following inequalities hold,

$$1 - \epsilon < |(u,v) - (0,1)| < 1 + \epsilon, \quad \forall (u,v) \in p_n(C_{local}), \forall n.$$

The rest of the curve, $C_{rest} = C \setminus C_{local}$, is compact and belongs to the upper half plane. By the uniform convergence (17), for each $\epsilon > 0$ we can choose n so that all points of $p_n(C_{rest})$ lie within the distance ϵ from C_0, cf. Figure 2. Hence

$$\lim_{n\to\infty} \Big(\max_{z_1 \in p_n(C)} \min_{z_2 \in C_0} ||z_1 - z_2|| + \max_{z_1 \in C_0} \min_{z_2 \in p_n(C)} ||z_1 - z_2|| \Big) = 0 \qquad (19)$$

By this, one has control on the first two terms in the definition of $\tilde{d}$. A consequence that will be used below, is that $\lim_{n\to\infty} A(p_n(C)) = \pi$.

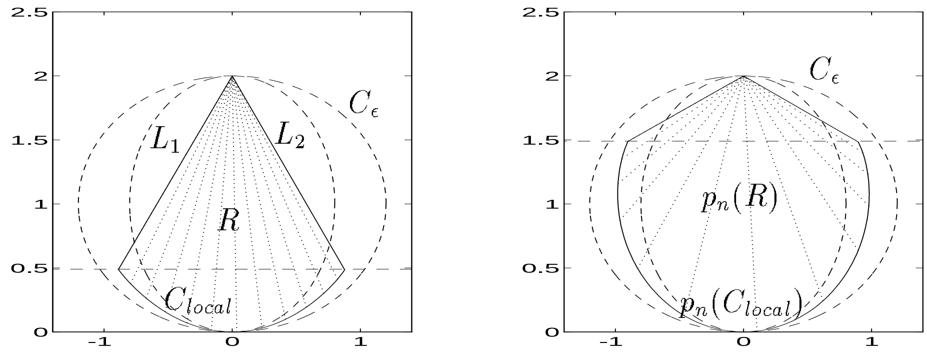

Fig. 3. The local part C_{local} together with two line segments form the boundary of a convex region R. The ellipse C_ϵ circumvents the convex region $p_n(R)$ for all n. For every n the transformed region $p_n(R)$ is convex and belongs to the interior of the ellipse C_ϵ.

It remains to consider the third term in $\tilde{d}$. The curve C is smooth around $(0,0)$, so it is possible to choose C_{local} so small that together with the lines L_1 and L_2 from the endpoints of C_{local} to $(0,2)$, it forms the boundary of a convex region R, cf. Figure 3. Since the shortest path circumventing a bounded region is the boundary of its convex hull, and since $p_n(C_{local})$ is part of the boundary of the convex region $p_n(R)$, we can deduce that $l(p_n(C_{local})) < l(C_\epsilon)$ for all n. By comparison with a circle of radius $1+\epsilon$ we get $l(C_\epsilon) < 2\pi(1+\epsilon)$. Since C_{rest} lies in a compact subset of the open upper half plane, by means of (18) we have

$$\limsup_{n\to\infty} l(p_n(C)) \le \limsup_{n\to\infty} l(p_n(C_{local})) + \limsup_{n\to\infty} l(p_n(C_{rest})) \le 2\pi(1+\epsilon) + 0.$$

Hence $\limsup_{n\to\infty} l(p_n(C)) \le 2\pi$. Since $l(p_n(C))^2 / A(p_n(C)) \ge 4\pi$, it follows that $\liminf_{n\to\infty} l(p_n(C)) \ge 2\pi$. Hence $\lim_{n\to\infty} l(p_n(C)) = 2\pi$, which concludes the proof.

An immediate corollary is

Corollary 3.

$$C \in \widetilde{\mathcal{C}} \Longrightarrow \inf_{p \in P_C} \frac{l(p(C))^2}{A(p(C))} = 4\pi$$

It has been proposed, e.g. in [BS1], to base a canonical representation $\bar{p}(C)$ of the curve C on the transformation $\bar{p}$ that minimizes the inverse compactness measure $l(p(C))^2/A(p(C))$. According to the corollary, the minimum is not attained if $C \in \widetilde{\mathcal{C}}$. This canonical representation is thus only well defined for curves that do not have a smooth and curved part on the convex hull, e.g. for polygons. It is, however, still possible that the local minima could be used, even for curves in $\widetilde{\mathcal{C}}$.

5 Non-Existence of Continuous Projective Invariants

In the proof of Theorem 2 one notices that the main part of the curve is squeezed into a neighbourhood of a point. For large n, the curve $p_n(C)$ looks like a circle, but has a small ripple that corresponds to the main part of the curve C. It turns out that if we slightly perturb the curve $p_n(C)$ outside this ripple and then do the inverse projective transformation, the new curve is almost identical to the original one. A consequence is the following somewhat surprising theorem.

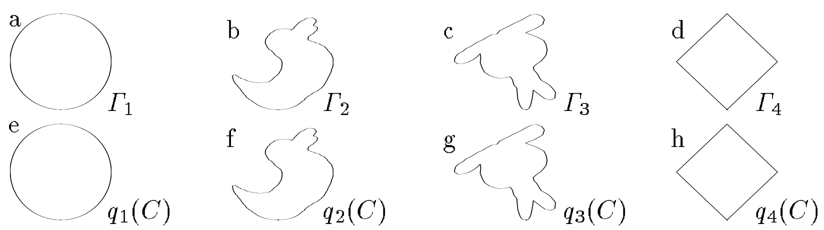

Fig. 4. The four upper curves are not projectively equivalent, but the four lower ones are.

Theorem 4. *Given $\Gamma_1, \ldots, \Gamma_m \in \mathcal{C}$. To every $\epsilon > 0$, there exist a curve C and projective transformations $q_1, \ldots, q_m$ so that*

$$\tilde{d}(q_i(C), \Gamma_i) < \epsilon, \qquad i = 1, \ldots, m.$$

The theorem is illustrated in Figure 4. Note that the curves Γ_i do not have to be smooth.

Proof. Since there is a smooth curve arbitrarily close to every curve fulfilling the assumptions above, it is no restriction to assume that the curves $\Gamma_1, \dots, \Gamma_m$ are smooth and therefore in $\tilde{\mathcal{C}}$. To each Γ_j associate a point P_j on C_0, a rectangle D_j and a sector S_j, according to Figure 5, where the sectors S_j are supposed to be pairwise disjoint.

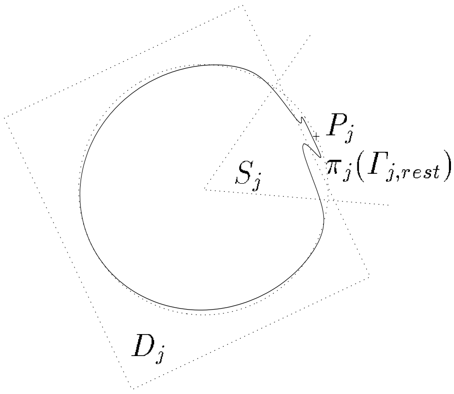

Fig. 5. Illustration of $\pi_j(\Gamma_j)$. After applying a projective transformation π_j to the curve Γ_j, the part in sector S_j will be glued with other corresponding parts to form a curve that approximates $\pi_j(\Gamma_j)$ for every $j = 1, \dots, m$.

Now the construction in the proof of Theorem 2 is used to cut each curve Γ_j into two pieces $\Gamma_{j,local}$, and $\Gamma_{j,rest}$, so that $\Gamma_{j,local}$ has arclength less than $\epsilon/2$. Hence all points of $\Gamma_{j,local}$ is at most a distance $\epsilon/4$ from a point in $\Gamma_{j,rest}$. It is possible to find projective transformations π_j such that $\pi_j(\Gamma_j)$ is at most $1/m$ from C_0 in the $\tilde{d}$-metric, and

$$\pi_j(\Gamma_{j,local}) \subset D_j \tag{20}$$

$$\pi_j(\Gamma_{j,rest}) \subset (\cap_{i \neq j} D_i) \cap S_j \tag{21}$$

$$l(\pi_j(\Gamma_{j,rest})) < 1/m \tag{22}$$

Since the inverse projection $q_j = \pi_j^{-1}$ has the contractive property (18), it can be chosen so that the it shrinks all curves in D_j of arclength less than a constant M, which will be specified later, into a curve with arclength less than $\epsilon/2$.

Let C be constructed by gluing the patches $\pi_j(\Gamma_{j,rest})$, the line segments obtained by radially connecting the endpoints of $\pi_j(\Gamma_{j,rest})$ with C_0, for all j, and the intermediate arcs of C_0. Both $C \setminus \pi_j(\Gamma_{j,rest})$ and $\pi_j(\Gamma_{j,local})$ are in D_j. Since C is a patch of m curves each with arclength less than $1/m$, of parts of the unit circle, and of $2m$ radial line segments of length less than $1/m$, the total arclength of $C \setminus \pi_j(\Gamma_{j,rest})$ is certainly less than $M = 1 + 2\pi + 2$. By the choice of q_j, this means that the curve $q_j(C \setminus \pi_j(\Gamma_{j,rest}))$ has arclength less than $\epsilon/2$. It is then clear that

$$\tilde{d}(q_j(C \setminus \pi_j(\Gamma_{j,rest})), \Gamma_{j,local}) < \epsilon.$$

The remaining part of C is $\pi_j(\Gamma_{j,rest})$, which is mapped identically into $\Gamma_{j,rest}$ by q_j. Hence $\tilde{d}(q_j(C), \Gamma_j) < \epsilon$.

The construction of C and q_i in the proof can be done by explicit formulas. Note that the transformations q_j are physically realisable in the pinhole camera model. An algorithm based on the proof have been implemented in MATLAB. Figure 4 has been constructed using this algorithm. Figure 6 shows what the mixed curve C looks like from eight different viewpoints. Observe that these eight different views are all projectively equivalent. Notice the kind of extreme, but non-singular, projective transformations that are involved.

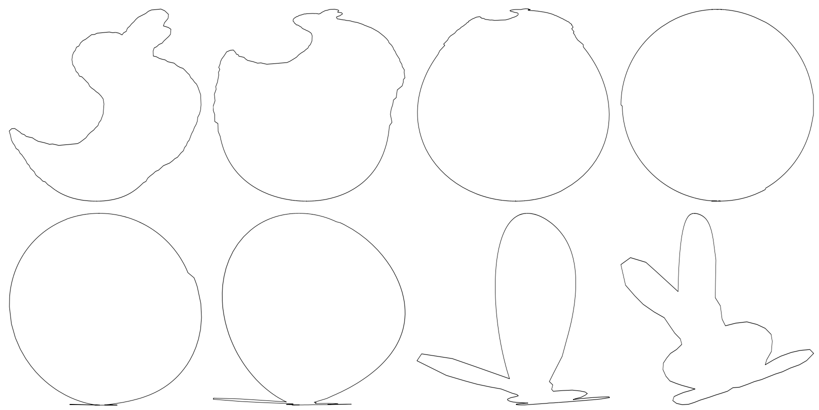

Fig. 6. Eight projectively equivalent views of the same planar curve. The duck transforms into a circle and then into a rabbit.

The theorem is in itself somewhat surprising and unintuitive at first, but it is a simple trick of hiding a shape along the convex hull of another shape. The reason it works is the use of extreme, but non-singular, projective transformations. The consequences are perhaps more important.

Corollary 5. *Let T be a projectively invariant mapping from the set of closed continuous curves with finite arc-length to a Hausdorff topological feature space, e.g. the real line, then T maps all curves at which it is continuous onto the same value.*

Proof. Assume to the contrary that $r_1 = T(\Gamma_1) \neq r_2 = T(\Gamma_2)$. Since the feature space is Hausdorff it is possible to find disjoint open sets $O_1 \ni r_1$ and $O_2 \ni r_2$. According to Theorem 4 the inverse images $T^{-1}(O_1)$ and $T^{-1}(O_2)$, which are open sets around Γ_1 and Γ_2, contain a projectively equivalent pair of curves, contradicting the assumption.

Suppose that we have a continuous projective normalization scheme that gives a unique representative from each equivalence class. It would then be possible to construct a continuous and non-constant projective invariant mapping from the set of curves with the Hausdorff metric to the real line. This is impossible according to Corollary 5. The conclusion is that *projective normalization schemes on the set of planar curves cannot both be continuous and give a unique representative from each equivalence class.* Either continuity or uniqueness has to be sacrificed.

6 Projective Normalization

Global projective invariants of curves are tricky. No matter what method you use, distinguished points, fitting ellipses, moments, projective smoothing or maximum compactness, you get non-uniqueness or discontinuity. In this section a normalization scheme is presented which might be useful. Only physically realisable transformations will be considered. Choose the normal reference frames according to

$$\Omega_P = \{\omega | \quad m_0(\omega) = 1, m_2(\omega) = aI, m_3(\omega) = 0, a \in R\} \tag{23}$$

This gives a normalization scheme with several representatives from each equivalence class. One way of locking the rotation is to demand that the maximum distance of a point in ω to the origin occurs at the x_1-axis. This method can be used also with convex curves.

The normalization scheme has been implemented and an experimental session will be presented. In this experiment, gray-scale images of roughly planar objects are taken with a digital camera. Polygon approximations of contours in the image are obtained using a Canny-Deriche edge detector. These curves are then normalized according to the proposed method, see Figure 7. Notice the good performance in Figures 7b and 7c. The three normalized curves lie practically on top of each other in spite of possible nonlinearities in the camera, errors in segmentation and in edge detection. The rabbit covered roughly 150×200 pixels in the image.

7 Discussion

Theorem 2 states that every smooth and closed rectifiable curve can be transformed arbitrarily close to a circle by means of a projective transformation,

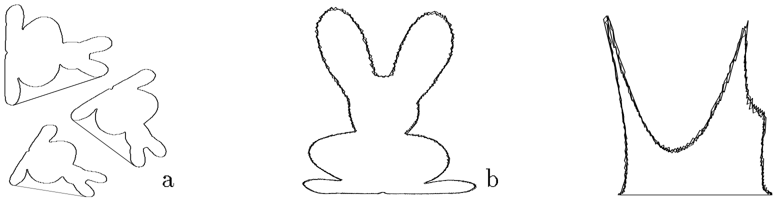

Fig. 7. Three images of rabbits, cf. Figure 7a, are normalized into a reference frame in which their third moment tensor is zero and the second moment matrix is the identity, cf. Figure 7b The same normalization is applied to the three regions enclosed by a bitangent and part of a contour, cf. Figure 7c

which is not affine. This fact does not per se have any implications on invariants, but we draw the conclusion that one should be careful when maximizing compactness over projective transformations.

The argument is similar to the trivial observation that one should not try to normalize curves under similarity transformations by minimizing area enclosed by the curve, since every rectifiable curve can be transformed arbitrarily close to a point simply by shrinking it.

It is also easy to see that every rectifiable curve can be transformed arbitrarily close in the Hausdorff metric to a line segment by means of an affine transformation, but we know from Theorem 1 that there are continuous affine invariant features of closed rectifiable curves.

Theorem 4 on the other hand has no counterpart in the similarity or affine cases. The reason is that affine transformations act in the same way around every point. It is not possible to contract one part of the curve and expand another. Given two curves of different affine shape it is not possible to construct a third curve and two affine transformations so both images of the third curve are arbitrarily close to the respective original.

A direct consequence of Theorem 4 is that there are no non-trivial stable projective invariants for closed planar curves. More precisely, every continuous and projectively invariant mapping from the set of closed rectifiable planar curves to the real line has to be constant. This means that a projective normalization scheme cannot both be continuous and give a unique representative from each projective equivalence class of curves. It is however possible that continuity can be achieved by sacrificing uniqueness as in Section 6, see also [5,8,11]. The perspective effects are usually small under normal viewing conditions. This fact should be possible to use in recognition algorithms.

The use of differential invariants has been proposed for recognition of planar curves. These are however very sensitive to noise. Therefore pre-smoothing of the curves is often necessary. Recently curve evolution schemes have been constructed that are both smoothing and invariant under similarity or affine transformations, cf. [ST1, ST2]. Projectively invariant curve evolution is currently under investigation. In light of Theorem 4 it is apparent that it is impossible to have a process that both commutes with projective transformations and smooths out small errors. When dealing with projective equivalence of planar curves it is impossible to discriminate between shapes on differents scales. What is small scale detail from one view-point is large scale shape from another.

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