Affine Invariants of Planar Sets

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Abstract

Recent research has indicated that invariants can be useful in computer vision for identification and pose determination of objects. The idea is to find functions that are invariant under a set of transformations acting on a configuration space. This paper describes some new viewpoints on the construction and use of such invariants. The key idea is that any kind of features like derivatives, distinguished points or integrative features can be used to construct invariants. In a given viewing situation one should choose those features that are most stable. As examples, affine invariants for planar smooth curves, planar regions, and planar point configurations are given. The properties of the invariants are illustrated with experiments.

1 Introduction

Identification and pose determination of three-dimensional objects are key problems in computer vision. It is surprising that even in rather complicated situations it is possible to solve them using only one single two dimensional view and a database of possible objects. One solution is to transform all the modeled objects using 'all' possible transformations according to some camera model and to compare the results with the actual view. This is an untractable method because it requires extensive computations. A more elegant way is to use features (numerical or descriptive) that are invariant under some group of transformation, usually affine or projective. A considerable amount of work has been done in this direction, cf. Mundy, Zisserman [13]. It easy to see that, unless restrictions are made on the class of objects, only trivial viewpoint invariants exist cf. e.g. [5, 19]. However, invariants do exist for planar objects. For planar point configurations affine and projective invariants have been used for model based recognition by e.g. [12, 11]. Invariant features for algebraic curves have also been used, cf. [7]. For nonalgebraic smooth curves several approaches have been tried. One idea is to construct invariants based on local differential properties, cf. [17]. This approach can handle any type of occlusion but has severe numerical difficulties because it is based on computations of derivatives. Another idea is to use distinguished points, such as bitangents, inflections and corners. The problem can then be reduced to point configuration invariants as discussed in [11]. The distinguished points can also be used to transform the curve into a canonical frame as was done in [15]. A third idea is to use both distinguished points and differential properties to construct semi-differential invariants, cf. [9]. Finally, integral invariants can be constructed using ideas from Duda and Hart [6]. Many methods have been developed where curves are transformed into a distinguished frame in which some function of the curve is optimized, cf. [6, 1, 2, 3, 10, 18]

These invariants are constructed by first transforming the object into a distinguished frame using some features, and then selecting some features (numerical or descriptive) of the object in this distinguished frame. The main idea of this paper is that any feature variant under the transformation group can be used to construct invariants. Some examples will be given of affine invariants for planar curves, regions and point configurations, where the features used to define a distinguished reference frame are averages and integrals. This makes the invariant robust to image distorsions, but not to occlusions. The latter complication will only be discussed briefly.

The paper is organized as follows. Section 2 contains the main idea about the construction of invariants. In section 3 examples are given that illustrate how the idea can be used. Complete and robust invariants are constructed for planar non algebraic smooth curves using weak isotropy. Moments are used to construct invariants for planar regions bounded by smooth curves. In Section 4 real experiments are presented together with an idea on how to deal with occlusion. Section 5 contains conclusions and ideas for further work.

2 Invariants and Recognition

In this section we first introduce some notation. The main idea is then given together with some practical considerations.

2.1 Preliminaries

The notion of a group acting on a set is briefly presented. This group action defines equivalence classes of objects which are said to have the same shape. Invariants are seen as mappings that are constant on the equivalence classes.

A group G is said to act on a set Ω if there exists a mapping

$$(G, \Omega) \ni (g, \omega) \longrightarrow g(\omega) \in \Omega$$

with the following properties

$$1(\omega) = \omega, \qquad \forall \omega \in \Omega$$

and

$$g_1(g_2(\omega)) = (g_1 \times g_2)(\omega), \forall \omega \in \Omega, \qquad \forall g_1, g_2 \in G$$

The notation for group action is either $g\omega$ or $g(\omega)$.

Two elements ω_1 and ω_2 are said to have the same shape if $\omega_1 = g\omega_2$ for some transformation $g \in G$. This is an equivalence relation, because of the group structure of G. We write

$$\omega_1 \sim \omega_2 \iff \exists g \in G, \omega_1 = g\omega_2 \tag{1}$$

The equivalence relation divides Ω into disjoint equivalence classes. Denote the equivalence class containing ω by $G\omega = \{g\omega | g \in G\}$. This is also called the orbit of ω under the group action G.

Let $T : \Omega \longrightarrow W$ be a function defined on Ω with values in some feature set W. This function is called an **invariant** if

$$\omega_1 \sim \omega_2 \Longrightarrow T(\omega_1) = T(\omega_2) \tag{2}$$

An invariant is called **complete** if

$$\omega_1 \sim \omega_2 \Longleftrightarrow T(\omega_1) = T(\omega_2) \tag{3}$$

In computer vision invariants are used to determine the shape of an image under some transformation group. Completeness is an important property. For complete invariants no information is lost when going from ω to $T(\omega)$. Any function of $T(\omega)$ is also an invariant. Furthermore all invariant features of ω can be calculated from $T(\omega)$. Useful invariants should also be easy to compute and stable under 'small' distorsions in ω.

Example The trivial mapping $T \equiv 0$ is an invariant, but it cannot be used to discriminate between any elements of different shape. □

Example The mapping $T : \omega \longrightarrow G\omega$ is a complete invariant. It is however difficult to work with if the set $G\omega$ is large. □

2.2 The Main Idea

The goal of this paper is to construct complete invariants. The main idea is to construct the invariants not only by searching for properties that do not change under the group action, but also by searching for properties that change a lot under the transformations. These latter properties will enable us to select a small number of representatives from each equivalence class. These will be the invariants.

A typical property could be obtained in the following way. Take any mapping $\mu : \Omega \longrightarrow R^n$. This will in general not be an invariant. The element ω is said to have the property P_{μ_0} if $\mu(\omega) = \mu_0$. Assume that μ can be chosen so that at least one element from each equivalence class has the property P_{μ_0} and let $\Omega_P = \{\omega | \mu(\omega) = \mu_0\}$. The projection of ω onto Ω_P along the equivalence class $G\omega$,

$$T(\omega) = G\omega \cap \Omega_P \tag{4}$$

is a complete invariant, see Figure 1. The set $T(\omega)$ is the representative for the equivalence class $G(\omega)$. The element ω can be recovered from each element in $T(\omega)$ if one knows the corresponding element in

$$U(\omega) = \{g \in G | g^{-1}\omega \in \Omega_P\} \tag{5}$$

The set of transformations $U(\omega)$ is a subset of G, and each element in $T(\omega)$ can be obtained from an element in $U(\omega)$ by

$$\omega' \in T(\omega) \Longrightarrow \exists g \in U(\omega), \omega = g\omega'$$

It follows that if $U(\omega)$ only has one element, then $T(\omega)$ also has only one element.

Assume for simplicity that both $T(\omega)$ and $U(\omega)$ consist of one element. An element ω_1 can then be represented as

$$\omega_1 = g_1 \omega_1^{inv}$$

with $g_1 = U(\omega_1)$ and $\omega_1^{inv} = T(\omega_1)$. The element ω_1 is thus represented as a product of two factors, one

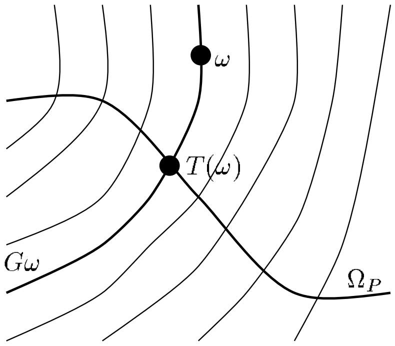

Figure 1: The group action divides the set Ω into disjunct equivalence classes. The invariant function T can be seen as the projection of an element ω along the equivalence class $G\omega$ onto Ω_P.

transformation and one element of Ω. One way to view this is to say that we have introduced an abstract coordinate system on $\Omega = G \times \Omega_P$. An element ω_1 has shape coordinate ω_1^{inv} and group coordinate g_1. Notice that ω_1^{inv} carries all information about the shape of ω_1 and that g_1 carries all information about where on the equivalence class ω_1 lies. A second element ω_2 is also represented as

$$\omega_2 = g_2 \omega_2^{inv}$$

with $g_2 = U(\omega_2)$ and $\omega_2^{inv} = T(\omega_2)$. These two elements have same shape if and only if $\omega_1^{inv} = \omega_2^{inv}$. If this is the case, then $\omega_1 = g_1 g_2^{-1} \omega_2$, as is seen in the following diagram.

$$\begin{array}{ccc} \omega_1 & & \omega_2 \\ \uparrow{\scriptstyle g_1} & & \uparrow{\scriptstyle g_2} \\ \omega_1^{inv} & = & \omega_2^{inv} \end{array}$$

In this way, we have developed a theoretical tool to determine the transformation between two elements of the same shape.

Remark. By the construction above the mapping $T(\omega) = G\omega \cap \Omega_P$ is an invariant for any choice of Ω_P as a subset of Ω. These invariants can be useless unless the set Ω_P is chosen carefully. A necessary and sufficient condition for T to be a complete invariant is that $\Omega_P \cap G\omega \neq \emptyset$ for all ω. □

As discussed earlier, useful invariants should be complete, continuous and easy to compute. These considerations give claims on the choice of Ω_P or alternatively on the features used to define Ω_P.

- $\Omega_P \cap G\omega \neq \emptyset, \quad \forall \omega \in \Omega$
- $\Omega_P \cap G\omega$ should be a small set.
- $\Omega_P \cap G\omega$ should be insensitive to distorsion in ω
- $\Omega_P \cap G\omega$ should be easy to compute

More attention should be given to the choice of features and how disturbances in the image affect them. This should give a better understanding on the statistical properties of the invariant features.

3 Affine Invariants for Planar Sets

The idea in Section 2 is general and works for any group acting on a set, e.g. planar projective transformations acting on a combination of contours, points, and regions. In this paper the idea will be used to construct affine invariants for planar curves, regions and finite point configurations.

To do this we first discuss the set Ω of configurations dealt with. These may appear both as objects and images. The affine imaging model is then introduced. The plane affine group has six degrees of freedom. These can be decomposed into translation (two d.o.f.), scaling (one d.o.f.), rotation (one d.o.f.) and shearing (two d.o.f.). Using an informal reasoning on the degrees of freedom, 6 property conditions are needed to construct invariants for the affine group. In this paper weak isotropy (Section 3.3) and moments (Section 3.4) will be used to define such conditions. The goal was to construct invariants for regions using features specific for regions and not use features based on distinguished points or the bounding curve.

In each case it will be shown that the set $U(\omega)$ is unique up to rotation. It then follows that $T(\omega)$ too is unique up to rotation. An additional property that fixes the rotation could be that the positive x_1-axis contains a point in the image with maximal distance to the origin. For most configurations there is only one point with maximum distance to the origin and for those the invariant set $T(\omega)$ contains only one element. In other cases $T(\omega)$ may contain a finite or even an infinite number of representatives, in which case the problem is inheritely difficult.

3.1 Objects and Images

It will be assumed that features are already detected in the image using some detector of edges or points of interest. The shape recognition routine proposed operates on these features. Typical configurations would then be composed of discrete points, curves and regions. It will be assumed that all objects are planar and that coordinate systems have been chosen both in the object and the image planes. This enables an identification of the planes so that object and image configurations can be regarded as elements of the same set Ω, consisting of subsets of R^2.

The configuration set Ω have different meaning in different sections.

- Ω is the set of piecewise smooth, non-linear curves in R^2 with finite arc length in Section 3.3
- Ω is the set of compact regions of R^2 with piecewise smooth boundary in Section 3.4

3.2 An Affine Imaging Model

The pinhole camera is a commonly used camera model which describes how objects in three dimensions are projected onto the image plane using a perspective transformation. If the relative depth in the scene is small, this planar perspective transformation can be approximated by an affine transformation. The examples in this paper deal with invariants under affine transformations.

As above, let object and image configurations be identified through the choices of coordinate systems. An affine transformation is a mapping from R^2 to itself given by

$$R^2 \ni x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + b = g(x) \in R^2$$

where A is a matrix and b a vector.

The determinant of A contains useful information about the transformation. If it vanishes, then the image collapses into a one dimensional set and information about the shape of the object is lost. To have a meaningful problem formulation the determinant must be non-zero. Moreover, there is no loss in generality to assume that it is positive. From now on the affine transformations with positive determinant will be called proper. The proper affine transformations form a group under composition. It acts on the set of images Ω according to

$$g(\omega) = \{g(x) = Ax + b | x \in \omega\}$$

which will be denoted as $g(\omega)$, $g\omega$ or $A\omega + b$.

The group of proper affine transformations has several subgroups.

$$\begin{array}{rl} A = I & \textit{the translation group } \mathcal{T} \\ A^T A = I, b = 0 & \textit{the group of rotations } \mathcal{O} \\ A^T A = I & \textit{the isometric group } \mathcal{C} \\ A^T A = (\det(A))^2 I & \textit{the similarity group } \mathcal{S} \end{array}$$

3.3 Planar Smooth Curves

In this section Ω will be the set of piecewise smooth, non-linear curves in R^2 with finite arc length. It will be assumed that the curves can be parametrized as

$$\omega = \left\{ \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \middle| \, t \in [t_0, t_1] \right\}$$

for some piecewise differentiable functions x_1 and x_2. The set G of possible transformations will be the group of proper affine transformations.

For each curve it is possible to calculate a relative distribution of directions $f(\alpha)$ such that $f(\alpha)d\alpha$ is the fraction of the arc length whose tangent orientation lies in the interval $[\alpha, \alpha + d\alpha]$. The curve is not oriented so the directions α and $\alpha + \pi$ are considered identical. Observe that the directional distribution is independent of the choice of parametrization. It is positive and normalized so that

$$\int_0^\pi f(\alpha) d\alpha = 1$$

The distribution $f(\alpha)$ is a π-periodic function which can be described by its Fourier series

$$f(\alpha) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{k \geq 1} A_k \cos(2k\alpha + d_k)$$

where A_k are positive amplitudes and d_k are phase shifts. These coefficients transform in a simple way when the curve is translated, rotated and scaled. A translation does not change the distribution at all, neither does a scale change. A rotation changes the phase shifts. A curve that is rotationally symmetric of order n has $A_k = 0$, when k is a multiple of n. If the curve is a circle then the distribution is constant and all A_i's are zero. This case when all directions are equally represented is called strongly isotropic. A weaker condition on isotropy has successfully been used by among others J. Gårding to determine the shape of a surface using surface markings, see [10, 18, 1]. Here we shall only use the definition and some of the basic properties.

Definition A curve is **weakly isotropic** if and only if $A_1 = 0$. □

The weak isotropy property is a good tool for constructing invariants of planar curves under affine transformations. The following theorem is essentially the result of [18]. A proof can be found in [20].

Theorem 1 *Given a curve $\omega \in \Omega$ there is a proper affine transformation unique up to rotations that* *transforms ω into a weakly isotropic curve ω' with mass center at the origin and mean distance one to the origin.*

An affine invariant for planar smooth curves can be constructed taking Ω_P as the planar non-linear smooth curves with finite arc length that (i) have mass centers at the origin, (ii) are weakly isotropic, (iii) have mean distance one to the origin and (iv) whose maximum distances from the origin occurs on the positive x_1-axis. According to Theorem 1 all parts of the affine transformation but the rotation are uniquely determined by properties (i-iii). For some images the rotation is not uniquely determined by the last property, in which case $U(\omega)$ and $T(\omega)$ consist of more than one element.

3.4 Planar Regions

In this section only closed contours or rather the regions bounded by closed contours will be considered. The set of configurations Ω are thus the set of regions in R^2 with positive measure, bounded by a piecewise smooth curve.

Definition Let the moments of a region ω be defined as

$$\begin{aligned} m_0(\omega) &= \int_{x\in\omega} dx \\ m_1(\omega) &= \int_{x\in\omega} x\, dx \\ m_2(\omega) &= \int_{x\in\omega} xx^T\, dx \end{aligned}$$

□

The moments change in a simple way when a region is transformed. For instance we have

$$\begin{aligned} m_1(\omega+b) &= m_1(\omega) + bm_0(\omega) \\ m_1(A\omega) &= |\det(A)|Am_1(\omega) \\ m_2(A\omega) &= Am_2(\omega)A^T|\det(A)| \end{aligned}$$

Therefore it is easy to use the moments to select representatives from each equivalence class.

Theorem 2 *Given a region ω, there is a proper affine transformation unique up to rotations that transforms ω into a region ω' with mass center at the origin and unitary second moment, i.e. $m_1(\omega') = 0$ and $m_2(\omega') = I$.*

Proof: The condition $m_1(\omega + b) = 0$ gives

$$m_1(\omega+b) = m_1(\omega) + bm_0(\omega) = 0$$

Since $m_0(\omega) \neq 0$, b is uniquely determined as

$$b = -\frac{m_1(\omega)}{m_0(\omega)}$$

Assuming that a region has $m_1(\omega) = 0$, A has to be chosen so that the second moment is the identity matrix. Observe that $m_1(\omega) = 0$ implies that $m_1(A\omega) = 0$, i.e. the mass center is not affected by A. The condition

$$m_2(A\omega) = |\det(A)|Am_2(\omega)A^T = I$$

gives

$$m_2(\omega) = |\det(B)|BB^T$$

with $B = A^{-1}$. Observe that $\det(m_2(\omega)) = \det(B)^4$. Let $|B|$ be the positive definite square root of the positive definite matrix BB^T, i.e. $|B|^2 = BB^T$. Then

$$|B| = \frac{\sqrt{m_2(\omega)}}{|\det(m_2(\omega))|^{1/2}}$$

The matrix A is thus given as

$$A = |B|^{-1}$$

It is uniquely determined up to rotations. ■

Remark. Somewhat cleaner formulas can be obtained by requiring $m_2(\omega)/m_0(\omega) = I$ instead of $m_2(\omega) = I$. The matrix $|B|$ is then given by

$$|B| = \sqrt{m_2(\omega)/m_0(\omega)}$$

□

To summarize, an affine invariant for planar regions can be constructed taking Ω_P as the planar regions that (i) have mass centers at the origin, (ii) has the identity matrix as the second moment and (iii) whose maximum distances from the origin occurs at the positive x_1-axis. As is shown above all parts of the affine transformation but the rotation are uniquely determined by properties (i) and (ii). For some images the rotation is not uniquely determined by the property (iii), in which case $U(\omega)$ and $T(\omega)$ consist of more than one element.

4 Experiments

4.1 Recognizing Animals

In order to test the ideas on real pictures and see to what extent the affine approximation holds, some experiments ware performed. Picture of animals were drawn with a dark green pen on a yellow paper, to make it easy for the edge-detectors. These animals were viewed without occlusion. A typical view can be seen in Figure 2. Edges were extracted using a Canny-Deriche edge detector (see Figure 3). The

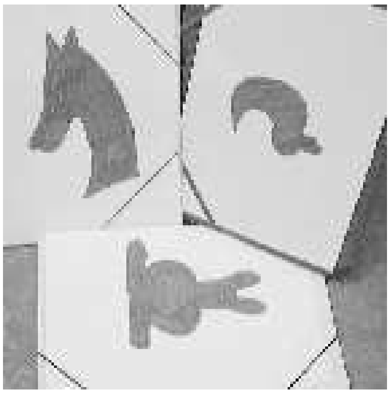

Figure 2: Digitized image of a planar horse, duck and rabbit.

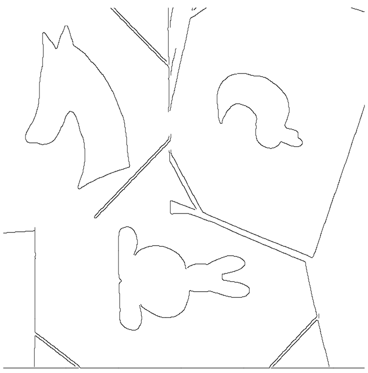

Figure 3: Edges from digitized image detected using Canny-Deriche edge detector.

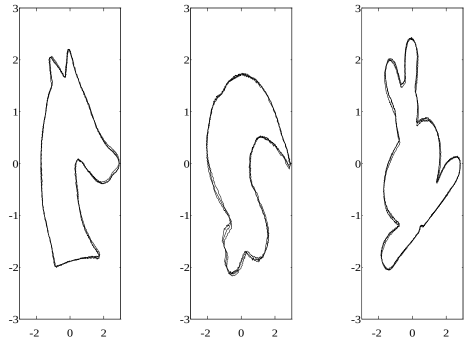

Figure 4: Shape coordinate for nine pictures, three each of a horse, a duck and a rabbit.

invariant mapping from Section 3.4 was used on all closed contours in the picture. Figure 4 shows the invariant values of three different views of the three objects, horse, duck and rabbit. Notice the good performance even though only affine invariants are used. Recall that this invariant is based only on the region enclosed by the closed contour, and is therefore insensitive to small disturbances of the contour. Features based on the contour itself or specific points are likely to be more sensitive to such errors. When extracting features from the invariant values in Figure 4, it is highly desirable that this is done using the same idea, i. e. by working with the regions rather than the curve itself. One idea is to use moments of other orders like 0, 3 or 4. Another idea is to calculate the area of a sector of the region versus the angle of sight from the center of mass. Let the sector region $R(\phi)$ of R^2 be

$$R(\phi) = \{(r\cos(\alpha), r\sin(\alpha))| \quad r > 0, \quad 0 < \alpha < \phi\}$$

and let the area of a sector of the region versus the angle of sight from the center of mass. function $A_\omega(\phi)$ be defined by

$$A_\omega(\phi) = \int_{R(\phi)\cap\omega} dx$$

If ω is star shaped or convex, no information is lost when going from ω to $A_\omega(\phi)$. For such regions we have the following relation.

$$A'_\omega(\phi) = r(\phi)^2/2$$

where $r(\phi)$ is the radius of the region at angle ϕ. For a general region $A_\omega(\phi)$ does not contain all information

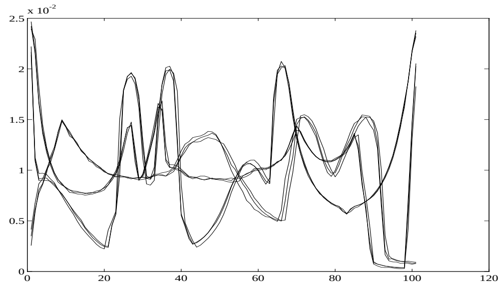

Figure 5: Invariant features for nine images, three of each animal.

about the original curve but it is still something that can be used to discriminate between different objects. In the experiment $n = 101$ bins were used. Each bin represents one sector and contains

$$a(k) = A(\frac{k+1/2}{2n\pi}) - A(\frac{k-1/2}{2n\pi})$$

These n features are then used to discriminate between different shapes. Figure 5 shows these invariant features for the three animals. Notice as before that only affine invariants are used.

4.2 Handling Occlusion

One major difficulty that has not been discussed yet is occlusion. Even though invariants based on integration over a region or a curve is insensitive to image distorsions, they are very sensitive to occlusions. One way to get around this is to use distinguished points to select a part of an curve or a region, and then calculate the invariant features based on this part. One algorithm for planar curves is as follows.

Algorithm 1

1 Extract a curve in the image.

2 Detect bitangents to that curve.

3 Let ω be the region bounded by the curve and the bitangent.

4 Transform ω into ω^{inv} with the following properties.

- $m_2(\omega^{inv}) = I$
- $m_1(\omega^{inv}) = 0$
- Bitangent of ω^{inv} is under the curve and parallel with the x_1 axis

5 Calculate the invariant features $(a(1), \ldots, a(n))$.

6 Use these features to index an array of models.

□

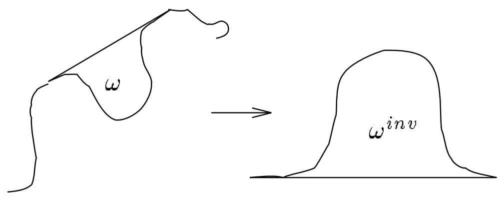

Figure 6: Illustration of Algorithm 1.

The algorithm is illustrated in Figure 6. This scheme has several good properties. The bitangent and therefore also the region ω is stable with respect to small disturbances of the curve, even though the two bitangent points are not well localized in the direction of the bitangent. The invariant representative is unique since the rotation is uniquely determined by the direction of the bitangent. The invariant features extracted are only based on the region and should therefore be quite robust. The properties of this algorithm have not yet been evaluated experimentally.

5 Conclusions and future experiments

This paper has proposed some new methods for object recognition in computer vision. Affine invariants for planar curves and planar regions are given. Practical experiments that show their insensitivity to image distorsions have been presented. This paper has focused almost entirely on the algebraic properties.

The work can be extended in several directions. It would be interesting to introduce a topology on Ω, to be able to discuss continuity of the invariants. In the experiments we have only compared the results visually. It would then be possible to formulate criteria, and search for optimal invariants. Very simple distorsion models were used. It would be interesting to incorporate also probabilistic models for image distorsions.

More attention should be given to the choice of features and how disturbances in the image affect them. This should give a better understanding on the statistical properties of the invariant features.

The algebraic aspects can also be pursued further. It is natural to try to apply similar ideas to invariants under projective transformations.

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References

[1] Blake, A. and Marinos, C., Shape from Texture: Estimation, Isotropy and Moments, *Artificial Intelligence, Vol 45, p. 332-380,* 1990.

[2] Blake, A. and Sinclair, D., On the projective normalisation of planar shape, *Technical Report OUEL, Oxford, Great Britain,* 1992.

[3] Brady, M. and Yuille, A., An Extremum Principle for Shape from Contour, *PAMI-6, No 3, p. 288-301,* 1984.

[4] Burkhardt, H., Fenske, A. and Schulz-Mirbach, H., Invariants for the Recognition of Planar Contour and Gray-scale Images, *Proc. of Workshop on Invariants, Genova, Italy,* 1992.

[5] Burns J. B., Weiss R., Riseman E. M., View Variation of Point-set and Line-segment Features, *Proc. of DARPA Image Understanding Workshop, Pittsburgh, Pennsylvania, USA, pages 650-659* 1990.

[6] Duda, R. O. and Hart, P. E., Pattern Classification and Scene Analysis, *Wiley-Interscience,* 1973.

[7] Forsyth, D., Mundy, J. L., Zisserman A., Invariant Descriptors for 3D Object Recognition and Pose, *ESPRIT/DARPA Invariants Workshop, Reykjavik, Iceland, pages 171-208* 1991.

[8] Golub, G. H. and Van Loan C. F., Matrix Computations, *The Johns Hopkins University Press, Baltimore, Maryland, USA* 1983.

[9] Van Gool, L., Moons, T., Pauwels, E. and Oosterlinck, A., Semi-differential Invariants, *ESPRIT/-DARPA Invariants Workshop, Reykjavik, Iceland, pages 359-386* 1991.

[10] Gårding, J., Shape from Surface Markings, *Ph.D. thesis, Dept. of Numerical Analysis and Computer Science, Royal Institute of Technology, Stockholm,* 1991.

[11] Huttenlocher, D. P. and Ullman, S., Object Recognition Using Alignment, *Proc. of ICCV1, pages 102-111* 1987.

[12] Lamdan, Y., Schwartz, J. T., and Wolfson, H. J., Affine Invariant Model-based Object Recognition, *IEEE Journal of Robotics and Automation, 6:578-589,* 1990.

[13] Mundy, J. L., and Zisserman A. (editors), Geometric invariance in Computer Vision, *to appear in MIT Press, Cambridge Ma, USA,* 1992.

[14] Nielsen, L., and Sparr, G., Projective Area-invariants as an Extension of the Cross-ratio, *Computer Vision, Graphics and Image Processing: Image Understanding, 54:145-159,* 1991.

[15] Rothwell, A., Canonical Frames for Planar Object Recognition, *Proc. of the Second European Conference of Computer Vision, Genova, Italy, pages 757-772* 1992.

[16] Sparr, G., Projective Invariants for Affine Shapes of point Configurations, *ESPRIT/DARPA Invariants Workshop, Reykjavik, Iceland, pages 151-170* 1991.

[17] Weiss, I., Noise-resistant Invariants of Curves, *ESPRIT/DARPA Invariants Workshop, Reykjavik, Iceland, pages 319-344* 1991.

[18] Witkin, A. P., Recovering Surface Shape and Orientation from Texture, *Artificial Intelligence, 17, p. 17-45* 1981.

[19] Åström, K., A Correspondence Problem in Laser-Guided Navigation, *Proc. Swedish Society for Automated Image Analysis, Uppsala, Sweden, pages 141-144,* 1992.

[20] Åström, K., Affine Invariants of Planar Sets, *Technical Report, Department of Mathematics, Lund Institute of Technology, Sweden,* 1992.