One-dimensional Retinae Vision

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Summary. In this chapter we investigate the geometry and algebra of multiple one-dimensional projections of a two-dimensional environment. This is relevant for one-dimensional cameras, e.g. as used in certain autonomous guided vehicles. It is also relevant for understanding the projection of lines in ordinary vision. A third application is for ordinary vision of vehicles undergoing so-called planar motion. The structure and motion problem for such cameras is studied, and all cases with non missing data are solved. For such problems we also characterize precisely when such problems have unique solutions. For cases with missing data the problem is considerably more difficult. Nevertheless, a classification of such problems is introduced, and some of the minimal cases are solved. Although the paper deals with calibrated cameras, it is shown that similar results exist for uncalibrated cameras. Some of the results also extend to ordinary vision.

1 Introduction

Understanding of one-dimensional cameras is important in several applications. In [21] it was shown that the structure and motion problem using line features in the special case of affine cameras can be reduced to the structure and motion problem for points in one dimension less, i.e. one-dimensional cameras. This was used to solve the problem of three views of seven lines. Two solutions were obtained. However, no geometrical interpretation of these two solutions were given.

Another area of application is vision for planar motion. It is shown that ordinary vision (two-dimensional retina) can be reduced to that of one-dimensional cameras if the motion is planar, i.e. if the camera is rotating and translating in one specific plane only, cf. [12]. In another paper the planar motion is used for auto calibration [2]. A typical example is the case where a camera is mounted on a vehicle that moves on a flat plane or flat road.

Our personal motivation, however, stems from **autonomous guided vehicles (AGV)**, which are important components for factory automation. Such vehicles have traditionally been guided by wires buried in the factory floor,

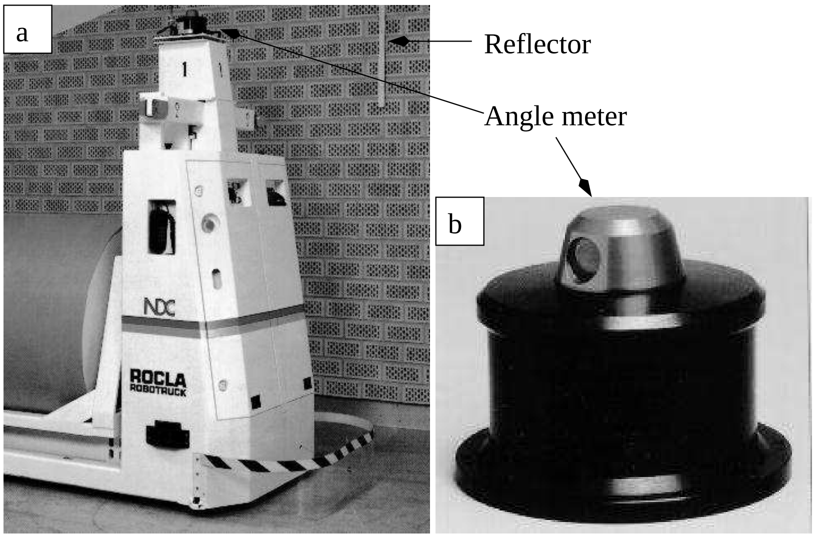

Fig. 1. a: A laser guided vehicle. b: A laser scanner or angle meter.

gives a very rigid system. Removal and change of wires is cumbersome and costly. The system can be drastically simplified using navigation methods based on laser sensors and computer vision algorithms. With such a system the position of the vehicle can be computed instantly. The vehicle can then be guided along any feasible path in the room. This paper deals with some navigation problems for laser guided vehicle **(LGV)**. The navigation system uses strips of inexpensive reflector tape (called **reflectors** or **beacons**) which are put on walls or objects along the route of the vehicle [15]. The **laser scanner**, also called the **angle meter** or **meter**, measures the direction from the vehicle to the beacons, but not the distance. This is the information used to calculate the position of the vehicle.

One interesting problem is the so-called **surveying** problem [3, 4]. This is the procedure to obtain a map of the unknown positions of the beacons using images at unknown positions and orientations. This is usually done off line, once and for all, when the system is installed, and then occasionally if there are changes in the environment. High accuracy is needed since the map has to be hard-coded in the system. The performance of the navigation routines depends on the precision of the surveyed map. The surveying problem is in essence a *structure and motion* problem, i.e. one tries to solve for both the structure (the map) and the motion of the vehicle.

Note that the discussion here is focused on finding initial estimates of structure and motion. In practice it is necessary to refine these estimates using non linear optimization or bundle adjustment [5, 23].

The chapter is a collection of results obtained together with Fredrik Kahl, Magnus Oskarsson and Niels Christian Overgaard. The chapter is organized as follows. In Sect. 2 a brief introduction to the geometry of the problem is given. Some important notations are introduced, and the structure and motion problem is formalized. In Sect. 3 we solve all structure and motion cases with non missing data. In Sect. 4 we classify similar problems with missing data. Some of the so-called prime problems are also solved. Both Sects. 3 and 4 assume that points and cameras are in general position. In Sect 5 we discuss cases in which there might be ambiguous solutions to the structure and motion problem without missing data.

2 Scanner Geometry

A laser navigated vehicle is shown in Fig. 1a The laser scanner, which is shown in detail in Fig. 1b, is mounted on the top of the vehicle. A laser beam generated by a vertical laser in the scanner is deflected by a rotating mirror at the top of the scanner. Thus, the laser beam scans the room at a fixed height. When the laser beam hits a beacon (a retroreflective tape, also shown in Fig 1a), a large part of the light is reflected back to the scanner. The reflected light is processed to find sharp intensity changes. When this happens the bearing α of the laser beam relative to a fixed direction of the scanner is stored. The time t when the reflection occurs is also stored. All beacons are identical. This means that the identity of a beacon cannot be determined from a single measurement.

We introduce an object coordinate system which will be held fixed with respect to the scene. The bearing α, defined above, depends on the position of the beacon (U_x, U_y) and the position (P_x, P_y) and orientation P_θ of the scanner, according to

$$\alpha(P, U) = \arg[U_x - P_x + i(U_y - P_y)] - P_\theta \,, \tag{1}$$

where arg is the complex argument (the angle of the vector $(U_x - P_x, U_y - P_y)$ relative to the positive x-axis). The vector (P_x, P_y, P_θ) is called the **camera state**.

Equation (1) for the measured bearing is non linear. A somewhat simpler representation of the same equation can be obtained as follows. The vector between the camera center and the beacon can be written as

$$\lambda \begin{bmatrix} \cos(\alpha + P_\theta) \\ \sin(\alpha + P_\theta) \end{bmatrix} = \begin{bmatrix} U_x - P_x \\ U_y - P_y \end{bmatrix} = \begin{bmatrix} 1 & 0 & -P_x \\ 0 & 1 & -P_y \end{bmatrix} \begin{bmatrix} U_x \\ U_y \\ 1 \end{bmatrix} . \tag{2}$$

By multiplying each side with a rotation matrix we obtain

$$\lambda \begin{bmatrix} \cos(\alpha) \\ \sin(\alpha) \end{bmatrix} = \begin{bmatrix} \cos(P_\theta) & \sin(P_\theta) \\ -\sin(P_\theta) & \cos(P_\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 & -P_x \\ 0 & 1 & -P_y \end{bmatrix} \begin{bmatrix} U_x \\ U_y \\ 1 \end{bmatrix} . \tag{3}$$

We introduce alternative representations for the **bearing**

$$\alpha \longleftrightarrow \mathbf{u} = \begin{bmatrix} \cos(\alpha) \\ \sin(\alpha) \end{bmatrix} \tag{4}$$

for the **beacon position**

$$(U_x, U_y) \longleftrightarrow \mathbf{U} = \begin{bmatrix} U_x \\ U_y \\ 1 \end{bmatrix} , \tag{5}$$

and for the **camera state**

$$(P_x, P_y, P_\theta) \longleftrightarrow \mathbf{P} = \begin{bmatrix} \cos(P_\theta) & \sin(P_\theta) \\ -\sin(P_\theta) & \cos(P_\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 & -P_x \\ 0 & 1 & -P_y \end{bmatrix} . \tag{6}$$

Using these notations, Eq. (1) can be written

$$\lambda \mathbf{u} = \mathbf{P}\mathbf{U} . \tag{7}$$

The alternative representation for the camera state above will be called the **camera matrix**. Notice that the structure of this 2×3 matrix is

$$\mathbf{P} = \begin{bmatrix} a & b & c \\ -b & a & d \end{bmatrix} , \tag{8}$$

with $a^2 + b^2 = 1$. It is straightforward to obtain the elements of the camera matrix from the meter state (P_x, P_y, P_θ) and vice versa.

It is sometimes useful to consider dual image coordinates

$$\alpha \longleftrightarrow \mathbf{v} = \begin{bmatrix} -\sin(\alpha) & \cos(\alpha) \end{bmatrix} , \tag{9}$$

so that $\mathbf{v}\mathbf{u} = 0$. This is particularly useful since it simplifies the camera constraint given by Eq. (7) to

$$\lambda \mathbf{v}\mathbf{u} = 0 = \mathbf{v}\mathbf{P}\mathbf{U} . \tag{10}$$

We will often use capital I to denote image number and capital J to denote point number. Thus $\mathbf{u}_{i,j}$ denotes the image direction for point J in image I, $\mathbf{P}_I$ denotes camera matrix for image I and $\mathbf{U}_J$ denotes object point number J.

3 Problem Formulation

Motivated by the previous sections the structure and motion problem will now be defined.

Problem 1. Given n bearings from m different positions

$$\mathbf{u}_{I,J}, \quad \forall (I,J) \in \mathbb{I} \tag{11}$$

the **surveying problem** is to find the depths $\lambda_{I,J} > 0$, the reconstructed points

$$\mathbf{U}_J = \begin{pmatrix} X_J \\ Y_J \\ 1 \end{pmatrix} \tag{12}$$

and the camera matrices

$$\mathbf{P}_I = \begin{pmatrix} a_I & b_I & c_I \\ -b_I & a_I & d_I \end{pmatrix}, \tag{13}$$

such that

$$\lambda_{I,J}\mathbf{u}_{I,J} = \mathbf{P}_I \mathbf{U}_J, \qquad (I,J) \in \mathbb{I}\,. \tag{14}$$

Here $\mathbb{I}$ is a subset of $\{1,\ldots,n\} \times \{1,\ldots,m\}$ that indicates which points are visible in which images. If all points are visible in all images, i.e. if $\mathbb{I} = \{1,\ldots,n\} \times \{1,\ldots,m\}$, we say that there is no missing data.

It is often convenient to consider things to be equal if they are equal up to scale. The notation $\sim$ will be used to denote equality up to scale. As an example two camera matrices $\mathbf{P}$ and $\widetilde{\mathbf{P}}$ are considered equal if $\mathbf{P} \sim \widetilde{\mathbf{P}}$. The reason for this is that $\mathbf{P}$ and $\widetilde{\mathbf{P}}$ give the same projections. Only the scale factor λ is different.

Definition 1. *The group of similarity transformations is defined as*

$$\mathcal{S} = \left\{ \mathbf{S} \sim \begin{pmatrix} e\cos(\theta) & -e\sin(\theta) & f \\ e\sin(\theta) & e\cos(\theta) & g \\ 0 & 0 & 1 \end{pmatrix} \right\}, \tag{15}$$

where θ denotes rotation, e change of scale and (f,g) translation.

We consider two solutions $(\lambda_{I,J}, \mathbf{U}_J, \mathbf{P}_I)$ and $(\widetilde{\lambda}_{I,J}, \widetilde{U}_J, \widetilde{P}_I)$ to the surveying problem to be the same if they are related by a similarity transformation. If there exists a transformation matrix $\mathbf{S}$ such that

$$\widetilde{\mathbf{U}}_J = \mathbf{S}\mathbf{U}_J\,,$$

$$\widetilde{\mathbf{P}}_I = \mu \mathbf{P}_I \mathbf{S}^{-1}\,,$$

$$\widetilde{\lambda}_{I,J} = \mu \lambda_{I,J}\,,$$

then both $(\lambda_{I,J}, \mathbf{U}_J, \mathbf{P}_I)$ and $(\widetilde{\lambda}_{I,J}, \widetilde{\mathbf{U}}_J, \widetilde{\mathbf{P}}_I)$ give the same projections $\mathbf{u}_{I,J}$, since

$$\lambda_{I,J}\mathbf{u}_{I,J} = \mathbf{P}_I \mathbf{U}_J, \qquad \forall \qquad (I,J) \in \mathbb{I}\,.$$

$$\widetilde{\lambda_{I,J}}\mathbf{u}_{I,J} = \widetilde{\mathbf{P}}_I \widetilde{\mathbf{U}}_J, \qquad \forall \qquad (I,J) \in \mathbb{I}\,.$$

In order to understand how much information is needed in order to solve the structure and motion problem, it is useful to calculate the number of degrees of freedom of the problem and the number of constraints given by the projection equation. Each object point has two degrees of freedom, and each camera state has three. The solution is only defined up to a similarity transformation, cf. Eq (15). This manifold $\mathcal{S}$ has dimension 4. Using n points and m cameras, we thus have $2n+3m-4$ degrees of freedom in the parameters. Each measured bearing gives one constraint on the estimated parameters. Assuming that each point is visible in every camera, we get mn constraints. The number of excess constraints $mn - (2n + 3m - 4)$ is given in Table 1. Disregarding the case of 1 point in 1 image, there are two interesting cases where the number of constraints is exactly equal to the number of degrees of freedom in the estimated parameters. These two cases

1. three images of five points $(m = 3, n = 5)$
2. four images of four points $(m = 4, n = 4)$

will be called the **minimal cases of the structure and motion problem**.

Table 1. The number of excess constraints $mn - (2n + 3m - 4)$ for the structure and motion problem with m images of n points.

	n						
m	1	2	3	4	5	6	7
1	0	−1	−2	−3	−4	−5	−6
2	−2	−2	−2	−2	−2	−2	−2
3	−4	−3	−2	−1	0	1	2
4	−6	−4	−2	0	2	4	6
5	−8	−5	−2	1	4	7	10
6	−10	−6	−2	2	6	10	14

4 Structure and Motion Problems Without Missing Data

4.1 Intersection and the Discrete Trilinear Constraint

In this section the simpler problem of determining the position of an object point using bearings from several known locations is studied. This problem is usually referred to as **intersection** or **reconstruction** in the literature [23].

The intersection problem for three bearings is connected to what is called the trilinear constraint. These trilinear constraints were originally developed for understanding of multiple view problems in ordinary vision [22, 24]. These constraints are interesting for several reasons.

First, they can be used to solve the more difficult surveying problem for three images. The relative motion of the cameras can be calculated from bearing measurements alone without calculating the structure of the object points explicitly. This gives a way to calculate an initial estimate of motion and of structure.

Second, the multilinear constraints can be used to eliminate faulty image correspondences, and to find new correct ones. This is essential for a robust, automatic structure and motion algorithm.

Only the calibrated case will be studied here because it is the natural situation for laser guided vehicles. Other camera models can be dealt with in a similar manner.

Problem 2. Given m bearing directions

$$\mathbf{u}_I, \quad I = 1, \ldots, m \tag{16}$$

from m known meters states

$$\mathbf{P}_I, \quad I = 1, \ldots, m \tag{17}$$

to one object point $\mathbf{U}$ in unknown position the **intersection problem** is to find the depths $\lambda_I > 0$ and the object point $\mathbf{U}$ such that

$$\lambda_I \mathbf{u}_I = \mathbf{P}_I \mathbf{U}, \qquad \forall\, I = 1, \ldots, m\,. \tag{18}$$

Each measured bearing from known position constrains the location of the object point to the line of sight. The equation for this line is easy to derive using dual image coordinates $\mathbf{v}$. Recall that

$$\underbrace{\mathbf{v}_I \mathbf{P}_I}_{\mathbf{l}_I} \mathbf{U} = 0\,, \tag{19}$$

thus $\mathbf{l}_I = \mathbf{v}_I \mathbf{P}_I$ is the line of sight. The geometric interpretation of the intersection problem is to intersect these m lines $(\mathbf{l}_1, \ldots, \mathbf{l}_m)$ at a point. The problem has no solution using only one measurement, but using two bearings the solution is, in general, unique.

4.2 The Calibrated Trilinear Tensor

The case of three cameras is of particular importance. Using three measured bearings from three different known locations, the object point is found by intersecting three lines. This is only possible if the three lines actually do intersect. This gives an additional constraint, which can be formulated in the following way

Theorem 1. *Let $\mathbf{u}_{1,J}$, $\mathbf{u}_{2,J}$ and $\mathbf{u}_{3,J}$ be the bearing directions to the same object point from three different camera states. Then the trilinear constraint*

$$\sum_{i,j,k} T_{i,j,k} \mathbf{u}^i_{1,J} \mathbf{u}^j_{2,J} \mathbf{u}^k_{3,J} = 0 \,, \tag{20}$$

is fulfilled for some $2 \times 2 \times 2$ *tensor* T.

Proof. By lining up the camera equations

$$\underbrace{\begin{pmatrix} \mathbf{P}_1 & \mathbf{u}_{1,J} & \mathbf{0} & \mathbf{0} \\ \mathbf{P}_2 & \mathbf{0} & \mathbf{u}_{2,J} & \mathbf{0} \\ \mathbf{P}_3 & \mathbf{0} & \mathbf{0} & \mathbf{u}_{3,J} \end{pmatrix}}_{M} \begin{pmatrix} \mathbf{U}_J \\ -\lambda_{1,J} \\ -\lambda_{2,J} \\ -\lambda_{3,J} \end{pmatrix} = \mathbf{0} \tag{21}$$

we see that the 6×6 matrix M has a non trivial rightnullspace. Therefore its determinant is zero. Since the determinant is linear in each column, it follows that it can be written as

$$\det M = \sum_{i,j,k} T_{i,j,k} \mathbf{u}^i_{1,J} \mathbf{u}^j_{2,J} \mathbf{u}^k_{3,J} = 0 \,, \tag{22}$$

for some $2 \times 2 \times 2$ tensor T.

The calibrated trilinear tensor $T = T_{i,j,k}$ in Eq. (20) will now be analyzed in more detail. Note that the constraint above only involves the *motion* parameters and the bearing directions. It does not involve the *structure* parameters $\mathbf{U}$. The tensor components can be calculated from the *motion* parameters. If we denote the rows of camera matrix $\mathbf{P}_I$ by $\mathbf{P}^1_I$ $\mathbf{P}^2_I$ it is straightforward to see that the tensor components are sub-determinants of the first three columns of the matrix M. In fact, the components can be obtained as

$$T_{ijk} = \wedge_{ii'} \wedge_{jj'} \wedge_{kk'} \det \begin{bmatrix} \mathbf{P}^{i'}_1 \\ \mathbf{P}^{j'}_2 \\ \mathbf{P}^{k'}_3 \end{bmatrix} , \tag{23}$$

where the tensor $\wedge$ is defined as

$$\wedge_{11} = 0, \qquad \wedge_{12} = -1, \qquad \wedge_{21} = 1, \qquad \wedge_{22} = 0 \ . \tag{24}$$

If the object coordinate system is changed

$$\mathbf{P}_1 \mapsto \widetilde{\mathbf{P}}_1 = \mathbf{P}_1\mathbf{S}, \quad \mathbf{P}_2 \mapsto \widetilde{\mathbf{P}}_2 = \mathbf{P}_2\mathbf{S}, \quad \mathbf{P}_3 \mapsto \widetilde{\mathbf{P}}_3 = \mathbf{P}_3\mathbf{S} \,, \tag{25}$$

where $\mathbf{S} \in \mathcal{S}$ denotes a 3×3 transformation matrix, the tensor components change according to

$$\begin{aligned} \widetilde{T}_{i,j,k} &= \wedge_{ii'} \wedge_{jj'} \wedge_{kk'} \det \begin{bmatrix} \widetilde{\mathbf{P}}^{i'}_1 \mathbf{S} \\ \widetilde{\mathbf{P}}^{j'}_2 \mathbf{S} \\ \widetilde{\mathbf{P}}^{k'}_3 \mathbf{S} \end{bmatrix} \\ &= \wedge_{ii'} \wedge_{jj'} \wedge_{kk'} \det \begin{bmatrix} \mathbf{P}^{i'}_1 \\ \mathbf{P}^{j'}_2 \\ \mathbf{P}^{k'}_3 \end{bmatrix} \det \mathbf{S} = (\det \mathbf{S}) T_{i,j,k} \,. \end{aligned} \tag{26}$$

A change of coordinate system only changes a common scale of the tensor.

It is natural to think of the tensor as being defined only up to scale. Two tensors T and $\widetilde{T}$ are considered equal if they differ only by a scale factor

$$T \sim \widetilde{T}\,. \tag{27}$$

Let $\mathcal{T}_u$ denote the set of equivalence classes of trilinear tensors.

As discussed in Sect. 2 only the relative motion of the camera is important.

Definition 2. *Let the manifold of* **relative orientation** *of three cameras be defined as the set of equivalence classes of three ordered camera matrices:*

$$\mathcal{P} = \left\{ (\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3) \,|\, \mathbf{P}_I = \begin{pmatrix} a_I & b_I & c_I \\ -b_I & a_I & d_I \end{pmatrix} \right\} \Big/ \simeq \tag{28}$$

where the equivalence is defined as

$$(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3) \simeq (\widetilde{\mathbf{P}}_1, \widetilde{\mathbf{P}}_2, \widetilde{\mathbf{P}}_3), \quad \exists\, \mathbf{S} \in \mathcal{S}, \widetilde{\mathbf{P}}_I \sim \mathbf{P}_I \mathbf{S}, I = 1,2,3\,. \tag{29}$$

Thus the above discussion states that the map $(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3) \mapsto T$ is in fact, a well-defined map from the manifold of equivalence classes $\mathcal{P}$ to $\mathcal{T}_u$.

It turns out that this mapping is in essence a two-to-one mapping. In fact, the following properties can be shown, [6].

Theorem 2. *A tensor $T_{i,j,k} \in \mathcal{T}_u$ is a calibrated trilinear tensor if and only if*

$$\begin{aligned} -T_{111} + T_{122} + T_{212} + T_{221} &= 0, \\ T_{112} + T_{121} + T_{211} - T_{222} &= 0\,. \end{aligned} \tag{30}$$

When these constraints are fulfilled it is possible to solve for the camera matrices in Eq. (23). There are, in general, two solutions possibly non real.

Corollary 1. *Let $\mathcal{T} \subset \mathcal{T}_u$ denote the submanifold fulfilling the constraint Eq. (30) then the map*

$$\begin{aligned} &T : \mathcal{P} \longrightarrow \mathcal{T} \\ &T(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3)_{ijk} = \wedge_{ii'} \wedge_{jj'} \wedge_{kk'} \det \begin{bmatrix} \mathbf{P}_1^{i'} \\ \bar{\mathbf{P}}_2^{j'} \\ \hat{\mathbf{P}}_3^{k'} \end{bmatrix} \end{aligned} \tag{31}$$

is a well-defined two-to-one mapping.

4.3 The Surveying Problem for Three Images

The previous section on the calibrated trilinear tensor provided us with the tool for solving the structure and motion problem for three cameras of at least five points.

Algorithm 1 Structure and motion from three images.

1. *Given three images of at least five points,*

$$\mathbf{u}_{I,J}, \quad I = 1, \ldots, 3,\ J = 1, \ldots, n, for n \geq 5\,.$$

2. *Calculate the trilinear tensor T that fulfills the linear constraints given in Eq. (30) and $\sum_{i,j,k} T_{i,j,k}\mathbf{u}^i_{1,J}\mathbf{u}^j_{2,J}\mathbf{u}^k_{3,J} = 0$, $\forall\, J = 1, \ldots n$.*
3. *Calculate the two possible solutions to the relative orientation $(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3)$ from T according to the proof of Theorem 2.*
4. *For each solution to the motion calculate structure using intersection.*

Note that five point correspondences give five linear constraints in Eq. (20). The fact that the camera is calibrated gives two additional constraints in Eq. (30). These seven constraints determine the eight components of T uniquely up to scale. Additional point correspondences will, in the ideal noise-free case, not give any additional constraints on T. There is thus a twofold ambiguity in the solution of the structure and motion problem, irrespective of the number of corresponding points. The calculations above do, however, not take the sign of the directions into account. Thus some of the reconstructed points sometimes have negative depth. This does not, however, guarantee uniqueness.

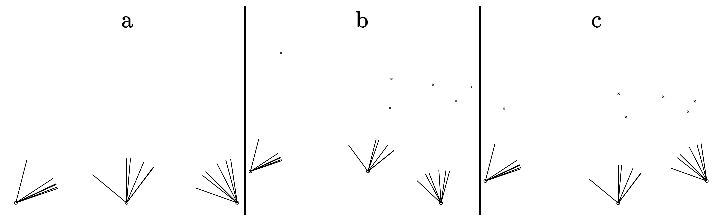

Fig. 2. **a**. The figure illustrates three images used as input in example 1. **b**. The first solution obtained from the analysis of the multilinear constraints. **c**. The second solution of structure and motion as obtained from the analysis of the multilinear constraints

Example 1. We illustrate the discussion above with a simple example. Fig. 2a shows three images of the same object points. Algorithm 1 is used to find the two possible solutions to the structure and motion problem T Fig. 2b,c. Note that in this example all points in both reconstructions have the correct orientation (positive depth). □

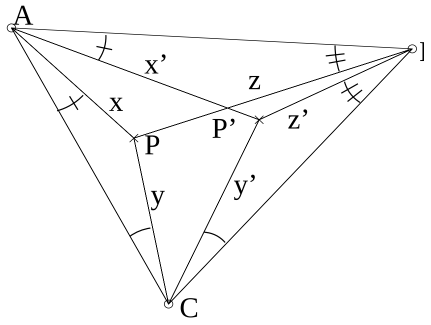

Fig. 3. A point P and its isogonal conjugate point P' with respect to triangle ABC

4.4 Understanding the Ambiguity

Looking at the solutions of the numerical examples (Fig. 2), one would like a geometric interpretation of the two possible solutions. It turns out that the ambiguity is a consequence of the following theorem about **isogonal conjugacy**, which is illustrated in Fig. 3 [7].

Theorem 3. *Let ABC be a triangle. Let x, y and z be lines through A, B and C respectively, that intersect in one point, say P. Let the line x' be the reflection of x in the bisector of the angle ABC, and similarly for y' and z'. Then the three lines x' y' and z' intersect at one point P'.*

A proof of the theorem is given in [11] and [1].

The points P and P' are called **isogonal conjugate points** with respect to the triangle ABC. An interesting property of such points is that they are focal points for a conic inscribed in the triangle, i.e. a conic that is tangent to all three sides of the triangle.

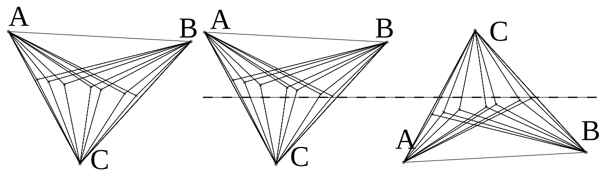

Fig. 4. The *leftmost picture* illustrates a triangle ABC, with drawn lines from the vertices to a number of points. The *centre picture* shows the same triangle, but using instead the corresponding isogonal conjugate points. The set of angles as seen from the three corners are the same but have different orientation. The *rightmost picture* is obtained from the centre picture by mirroring in the broken line. By turning the centre picture upside down the same angles are observed from the three corners, although the shape of points in the interior is different

Making the same constructions as in Fig. 3 for another pair of points Q and Q', we see that seen from each of the positions A, B and C, the absolute values of the angles between P, Q and P', Q' are equal. However, the orientation is wrong. To get the same orientation the construction needs to be turned up-side-down (or mirrored in an arbitrary line). This leads to the following corollary:

Corollary 2. *For every solution to the structure and motion problem for three cameras, another solution can be constructed by first changing the object positions to their isogonal conjugates with respect to the three camera positions, and then mirroring the camera and object positions in some line.*

An illustration is given in Fig. 4.

The other minimal case of four points in four images can be solved in a similar technique, by introducing a dual quadrilinear tensor. For more details on this, see [6]. This case is, however, *dual* to the case of three views of five points solved previously. This duality is described in the following section.

Example 2. Using the following bearing measurements

$\alpha_{I,J}$	J			
I	1	2	3	4
1	−2.3562	2.3562	1.1844	−0.7602
2	−2.1588	2.6779	0.7833	−0.9956
3	−2.5536	2.5536	1.7567	−1.1651
4	−2.3562	2.8198	2.0054	−1.3120

we obtain two possible solutions on the meter states and the object positions, which are illustrated in Fig. 5.

4.5 Duality Between Number of Points and Number of Images

Table 2 give the number of solutions in general to the surveying problem of m images of n points.

Table 2 has a symmetric appearance. This can be shown using a technique that Carlsson developed in [8, 9]. The proof that he did for uncalibrated projections from 3D to 2D can be used even in the case of uncalibrated projections from 2D to 1D:

Theorem 4 (Carlsson Duality). *The uncalibrated surveying problem with n points and m images is equivalent to the uncalibrated surveying problem with $m + 3$ points and $n - 3$ images.*

Proof. This is simplest seen by choosing the image coordinates of the first three points according to

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \mathbf{u}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{32}$$

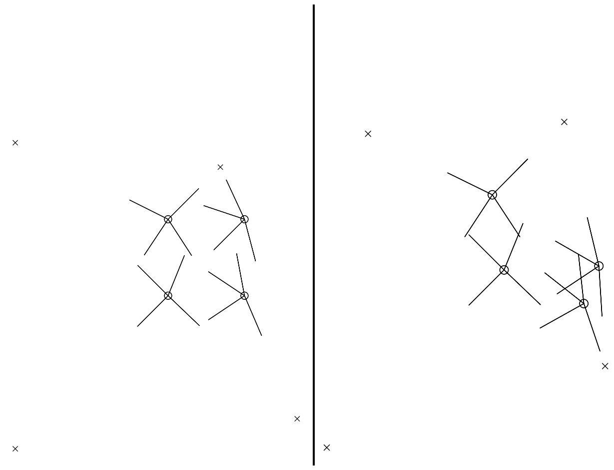

Fig. 5. Two different solutions to the structure and motion problem with 4 images of 4 points. The image directions are shown as unit vectors from the center of the camera. The object points are shown as small stars

Table 2. The number of solutions to the surveying problem with m images of n points. *Superscript stars* indicate overdetermined situations

m \ n	3	4	5	6	7
2	∞	∞	∞	∞	∞
3	∞	∞	2	2^*	2^*
4	∞	2	1^*	1^*	1^*
5	∞	2^*	1^*	1^*	1^*
6	∞	2^*	1^*	1^*	1^*

and the object coordinates of the first three points according to

$$\mathbf{U}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{U}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{U}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{33}$$

Since $\lambda_i \mathbf{u}_i = \mathbf{P}_i \mathbf{U}_i$: it follows that the camera matrix has the following form:

$$\mathbf{P} = \begin{pmatrix} V_1 & 0 & V_3 \\ 0 & V_2 & V_3 \end{pmatrix}.$$

The camera equation for the remaining points,

$$\mathbf{u}_{i,j} = \begin{pmatrix} U_1 V_1 + U_3 V_3 \\ U_2 V_2 + U_3 V_3 \end{pmatrix}, \tag{34}$$

is symmetric in camera parameters (V_1, V_2, V_3) and structure parameters (U_1, U_2, U_3). Thus any algorithm for solving n points in m images can be used to solve the $m+3$ points in $n-3$ images.

Theorem 5. *The calibrated surveying problem with n points and m images is equivalent to the calibrated surveying problem with $m+1$ points and $n-1$ images.*

Proof. This follows immediately from Theorems 6 and 4.

According to Theorem 5 the 4 points in 4 images problem is equivalent to the 5 points in 3 images-problem. This explains the symmetry in Table 2.

4.6 Connection to Uncalibrated Cameras

In Sect. 3.2 it was shown that the problem of 5 points in 3 images has in general two solutions. It was also shown that if there is a solution to the problem of more than 5 points in 3 images, then there are 2 solutions. Similarly in Sect. 3.4 it was shown that the problem of 4 points in 4 images has two solutions. If there is a solution to the problem of 4 points in more than 4 images then there are two solutions. The problem of at least 5 points in at least 4 images is overdetermined, and if there is a solution it is in general unique. The situation is illustrated in Table 2.

In this paper it has been assumed that bearings are measured and therefore that the camera matrix has the special form given by Eq. (8). In many situations it can be of interest to study the structure and motion problem for so-called uncalibrated cameras. This is identical to the surveying problem, except that the camera matrix is allowed to be a general 2×3 matrix. The difference in the study of minimal cases is, however, slight, due to the following theorem.

Theorem 6. *Knowing that the camera is corrected for internal calibration is equivalent to seeing two extra points (the circular points) in each image.*

Thus it follows that for the uncalibrated surveying problem, in the two minimal cases are three images of seven points and four images of six points. In both of these situations there is a twofold ambiguity in the solution. The ambiguity is not resolved by adding points in the 3 image case or adding images in the 6 point case.

4.7 Solution to all Cases with Nonmissing Data

If it is possible to solve a case with a subset of cameras and beacons, then it is often relatively easy to extend that solution to other cameras and points by resection and intersection. If all points are visible in all images then any well-defined case above can be solved by first solving for one of the two minimal cases with nonmissing data, i.e. 4 views of 4 beacons and 3 views of 5 beacons.

5 Structure and Motion Problems With Missing Data

The aim of this section is to solve all structure and motion problems for the case of missing data. Depending on the index set $\mathbb{I}$, which describes which points are visible in which images, a structure and motion problem can be either

- ill-defined, if there is not generally enough data to constrain all unknown variables
- well-defined and minimal, if there is exactly enough data to constrain the unknown variables (up to a discrete number of solutions)
- well-defined but overconstrained, if there is more than enough data to constrain the unknown variables

The goal of this section is to classify the possible index sets $\mathbb{I}$ into these three categories.

Some of the minimal cases contain a minimal case as a subproblem. An example of this is the case with 4 points seen in 5 images, but where the fourth point is missing from the fifth image. It is minimal, but contains a subproblem (the problem with the first 4 views only), which is well defined and minimal. We will use the notation **prime problem** for a minimal problem which does not contain a well defined minimal problem as a subproblem. A minimal but not prime problem may in some cases be solved by first solving the contained prime problem and then extending the solution using resection and intersection. In other cases the prime problem may be embedded in the minimal problem in a more complicated manner. We first observe that similar to the case of nonmissing data a well-defined but overconstrained problem contains as a subset a problem that is well-defined but minimal. Thus by finding the minimal cases and solving them, we should be able to solve all well-defined problems by the following algorithm:

1. Find whether a problem contains a well-defined minimal problem as a subset.
2. Solve the structure and motion problem for this subset.
3. Extend the solution to the original problem.

As the classification is based on the index set $\mathbb{I}$ alone, it is interesting to study these sets. In this paper we consider these sets as binary matrices, that is visibility matrices A of size $m \times n$, where black denotes missing data and white denotes a measurement that is present. Another way of viewing these index sets is as bipartite graphs with $m + n$ nodes. There is an edge between node I in the first set and node J in the second set if the point J is visible in image I. Thus a well-defined minimal case can be considered to be a subgraph of a well-defined but overconstrained problem. Here we will use the notation $|\mathbb{I}|$ to denote the number of elements in the set $\mathbb{I}$.

In the following two sections the problem of classifying structure and motion problems for 1d retina will be addressed. In Sect. 5.9 we will consider the classification of 2D retina problems.

5.1 Classification of Structure and Motion Problems

The goal of this section is to give some conditions on what constitutes a well-defined minimal problem. From these minimal problems the prime problems can be determined.

5.2 Equivalence Classes of Index Sets

The labeling of the cameras and of the beacons are of no consequence to the structure of the problem under study. Two index sets are considered equivalent if one results from the other by suitable relabelings. This means that there are many structure and motion problems that have different $\mathbb{I}$ but that correspond in principle to the same problem.

Definition 3. *An index set $\mathbb{I}$ is said to be of type (m, n, l) if it represents a situation with m images and n points, in which exactly l points are not visible in all of the images, that is, if $|\mathbb{I}| = mn - l$.*

From this definition it is clear that an index set $\mathbb{I}$ of type (m, n, l) can be represented by a binary $m \times n$ matrix $A = (a_{IJ})$ with $a_{IJ} = 1$ if $(I, J) \in \mathbb{I}$, and $a_{IJ} = 0$ otherwise, and such that $\sum_{IJ} a_{IJ} = mn - l$. The possible index sets of type (m, n, l) are thus in one-to-one correspondence with the set

$$M(m, n, l) = \{A \in \mathrm{Mat}_{m \times n}(\mathbb{Z}_2) : \textstyle\sum_{IJ} a_{IJ} = mn - l\}.$$

Let S_k denote the group of permutations on k symbols. With each permutation $\sigma \in S_k$ is associated a $k \times k$ permutation matrix, which will also be denoted by σ.

Definition 4. *Two $m \times n$ matrices A and B are said to be* permutation equivalent *if there exist permutations $\sigma \in S_m$ and $\tau \in S_n$ such that $B = \sigma^T A \tau$. If A and B are permutation equivalent then we write $A \sim B$.*

The notion of equivalence of index sets can now be given a formal definition

Definition 5. *Two index sets $\mathbb{I}$ and $\mathbb{I}'$ are called* equivalent *and we write $\mathbb{I} \sim \mathbb{I}'$ if their corresponding matrix representations are permutation equivalent.*

The relation $\sim$ is easily seen to be an equivalence relation. It follows that $M(m, n, l)$ (or the corresponding index sets) can be partitioned into equivalence classes $M_1, \ldots, M_\omega$ of matrices (or index sets). The number of essentially different index sets is thus seen to be exactly the same as the number $\omega = \omega(m, n, l)$ of equivalence classes. This is the number of principally different problems of type (m, n, l). The number ω also represents the number of different bipartite graphs with l edges from m to n nodes.

5.3 The Germs

A first characterization of a well defined minimal structure and motion problem is that it contains exactly the same number of equations as unknowns. If we concentrate on the case of calibrated cameras with 1D retina, then each object point has two degrees of freedom and each camera state has three. The solution is only defined up to a similarity transformation. This manifold has dimension 4. Using n points and m cameras we thus have $2n+3m-4$ degrees of freedom in the parameters. Each measured bearing gives one constraint on the estimated parameters. Thus for a problem with visibility index set $\mathbb{I}$ we have $|\mathbb{I}|$ equations. This means that minimal problems have $|\mathbb{I}| = 2n+3m-4$. Since the maximum number of equations with m views of n points is mn, it is easy to see how many measurements l that have to be occluded to obtain minimal problems, $l = mn - (2n+3m-4)$. This number is shown in Table 1.

In order to find the minimal problems we concentrate our efforts on problems of type $[m, n, mn - (2n+3m-4)]$.

Definition 6. *A structure and motion problem of type* $[m, n, mn-(2n+3m-4)]$ *is said to be a* germ *of a minimal problem.*

For a structure and motion problem to be minimal and/or prime the condition of being a germ is only a necessary condition.

5.4 The Prime Condition

For a given germ the corresponding structure and motion problem can be minimal or ill-posed. If it is minimal it may or may not be prime. The question of which class a germ belongs to can be categorized in terms of the graph of the index set. We will use the following intuitive assumption.

Conjecture 1. For a given germ with index set $\mathbb{I}$, the corresponding structure and motion problem is minimal iff no subgraph of $\mathbb{I}$ is overdetermined.

An empirical method for determining whether a problem is minimal and well defined is to calculate the Jacobian of the bundle adjustment problem and study its singular values. We have used this technique to empirically check our conjecture.

It is clear that if a subgraph of a germwith index set $\mathbb{I}$ is overdetermined then there has to be a part of the problem that is underdetermined, and hence the whole problem is ill-posed.

Theorem 7. *Given a germ with index set* $\mathbb{I}$*, at least one subgraph of* $\mathbb{I}$ *is overdetermined* $\Rightarrow$ *the corresponding structure and motion problem is ill-posed*

We will henceforth identify the class of minimal problems with those that fulfill Conjecture 1. Under this assumption the notion of being a prime problem can be given the following formal definition:

Definition 7. *A* prime problem *is a germ with index set* $\mathbb{I}$ *such that all strict subgraphs of* $\mathbb{I}$ *are underdetermined.*

A minimal problem that is not prime is an extension of a prime problem. The extended minimal problem can in many cases be solved by a succession of resections and intersections based on the solution to the prime case. In other cases the extension can be more complicated.

Definition 8. *An extension of type* (m, n) *is an extension with* m *extra cameras and* n *extra points of a prime problem. That is, the problem is extended by* m *additional cameras and* n *additional points, see e.g. Fig. 9.*

5.5 The Germ Investigation

We now concentrate our efforts on finding out how many germs there are for different numbers of cameras and points. From these germs we then determine which are minimal and which are prime.

5.6 Equivalence Classes of Germs

Let the type (m, n, l) be fixed throughout the remainder of the discussion. To compute $\omega = \omega(m, n, l)$ notions and results from group theory will be used. Our reference here is to Sect. 3.6 of Fraleigh's text [13].

First, denote the product group $S_m \times S_n$ by G. Second, if $g = (\sigma, \tau) \in G$ and $A \in M = M(m, n, l)$, then a group action of G on M is defined by the formula

$$g \cdot A = \sigma^T A \tau. \tag{35}$$

Thus two matrices $A, B \in M$ satisfy $A \sim B$ if and only if there exists $g \in G$ such that $g \cdot A = B$. The equivalence classes $M_1, \ldots, M_\omega$ of $\sim$ correspond to **the orbits in M under the action of G.** Therefore ω can be computed by the following well-known formula of Burnside: for any $g \in G$ let $M_g = \{A \in M : g \cdot A = A\}$ denote the set of matrices that are fix-points under action by g. Then

$$\omega = \frac{1}{|G|} \sum_{g \in G} |M_g|. \tag{36}$$

While Eq. (36) solves our problem in theory, there are still some practical problems to overcome. First, given $g \in G$, how do we compute $|M_g|$? Second, the sum $\sum_{g \in G} |M_g|$ must be evaluated, but as $|G| = m!n!$ becomes very large very quickly, the sheer size of G may become an obstacle, unless the evaluation is performed cleverly.

A permutation $g = (\sigma, \tau) \in G$ may be regarded as an element of S_{mn}, as $A \mapsto \sigma^T A \tau$ permutes the mn entries of A. Let $g = g_1 g_2 \cdots g_s$ be the factorization in S_{mn} of g into a product of commuting (or disjoint) cyclic permutations. It is now easy to see that $A \in M_g$ if and only if the entries in A, which equal zero, are arranged in such a manner that any cycle g_i is either completely occupied by entries equal to zero, or contains no such entry at all. It follows that $|M_g|$ equals the number of ways in which l zeros can be allocated to $m \times n$ entries, such that the condition just described is satisfied. It is clear from this discussion that $|M_g|$ only depends on g's cycle structure (the number of cycles and their lengths).

Definition 9. *If $\sigma \in S_k$ is a permutation in k symbols, let $n_i(\sigma), i = 1, \ldots, k$ denote the number of i-cycles in the factorization of σ into commuting cycles. The* cycle index of σ *is the polynomial*

$$P_\sigma(x_1, x_2, \ldots, x_k) = x_1^{n_1(\sigma)} x_2^{n_2(\sigma)} \cdots x_k^{n_k(\sigma)}. \tag{37}$$

If $H < S_k$ is a (sub)group of permutations, then the cycle index of H *is the polynomial*

$$P_H(x_1, x_2, \ldots, x_k) = |H|^{-1} \sum_{h \in H} P_h(x_1, x_2, \ldots, x_k).$$

It follows from the theory developed in [26] that

$$|M_g| = (l!)^{-1} (\mathrm{d}/\mathrm{d}x)^l P_g(1 + x, 1 + x^2, \ldots, 1 + x^{mn})|_{x=0},$$

for any $g \in G$. This formula solves the first of our two problems. Furthermore, it follows from Burnside's formula Eq. (36) that

$$\omega = \frac{1}{l!} \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^l P_G(1 + x, 1 + x^2, \ldots, 1 + x^{mn}) \Big|_{x=0}. \tag{38}$$

It turns out that the cycle index P_H is reasonably easy to compute when H is all of S_k. Now, $G = S_m \times S_n$ is a proper subgroup of S_{mn}, so in view of Eq. (38) our second problem above becomes: How do we compute P_G when the cycle indices of S_m and S_n are known? Again the authors of [26] provide the answer: they introduce a new operation beside the usual addition and multiplication, denoted $*$, on the ring of polynomials in infinitely many variables $x_1, x_2, x_3, \ldots$, and with rational coefficients. The "product" is associative, commutative and distributive over both $+$ and $\cdot$, so it suffices to describe $*$ on monomial factors x_i^m and x_j^n, in which case

$$x_i^m * x_j^n = x_{[i,j]}^{imjn/[i,j]}, \tag{39}$$

where $[i, j]$ is the least common multiple of i and j. The authors of [26] then proceed to prove the following beautiful result, which we have used to compute P_G:

Theorem 8 (Wei and Xu). *If $H < S_m$ and $K < S_n$ are (sub)groups, then $H \times K < S_{mn}$, and $P_{H \times K} = P_H * P_K$.*

Example The cycle index of S_3 is $\frac{1}{6}(x_1^3 + 3x_1x_2 + 2x_3)$ so if $G = S_3 \times S_3$ then

$$\begin{aligned} P_G =& \tfrac{1}{6}(x_1^3 + 3x_1x_2 + 2x_3) * \tfrac{1}{6}(x_1^3 + 3x_1x_2 + 2x_3) \\ =& \ \tfrac{1}{36}(x_1^9 + 6x_1^3x_2^3 + 9x_1x_2^4 + 12x_3x_6 + 8x_3^3), \end{aligned}$$

so

$$P_G(1+x,\dots,1+x^9) = 1 + x + 3x^2 + 6x^3 + 6x^6 + 7x^4 + 7x^5 + 3x^7 + x^8 + x^9.$$

It then follows from Eq. (38) that

$$\omega(3,3,3) = \frac{1}{3!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3 P_G(1+x,\dots,1+x^9)\ \Big|_{x=0} = 6.$$

The procedure for calculating ω, described above, was implemented in Maple (Maplesoft, Canada). Using this program we are able to compute ω for any given (m,n,l) and in particular for the germs. Table 3 contains ω for the first few types (m,n,l), with l given by Table 1.

5.7 Finding and Classifying Germs

We calculated the equivalence classes for some of the first germs using algorithms described in [20]. In table 3 the number of distinct germs for these cases are given. These germs where then classified as being minimal and possibly also prime. The numbers of such problems are shown in Tables 4 and 5.

Table 3. The number ω of different germs for different m and n

ω	n					
m	4	5	6	7	8	9
3	-	1	1	3	6	11
4	1	3	16	62	225	
5	1	16	155	1402		
6	3	79	1799			
7	6	361				
8	16					

In Fig. 6 and Fig. 7 the prime problems for the configurations of type $(5,5,4)$ and $(4,6,4)$ are given. The configurations inf Fig. 6a-c seem to be connected to configurations in Fig. 7a-c. The similarity can be explained by the Carlsson duality.

If one looks at Table 5, the number of prime configurations seem to increase quickly as both m and n increase. This leads to the question whether this is true or if the number of prime cases after some time stops growing. One can at least give the result in Theorem 9.

Table 4. The number of minimal configurations for different m and n

	n					
m	4	5	6	7	8	9
3	-	1	1	2	3	4
4	1	3	12	41	118	
5	1	12	110	876		
6	2	48	1050			
7	3	159				
8	5					

Table 5. The number of prime configurations for different m and n

	n					
m	4	5	6	7	8	9
3	-	1	0	0	0	0
4	1	1	3	5	8	
5	0	3	22	145		
6	0	6	136			
7	0	0				
8	0					

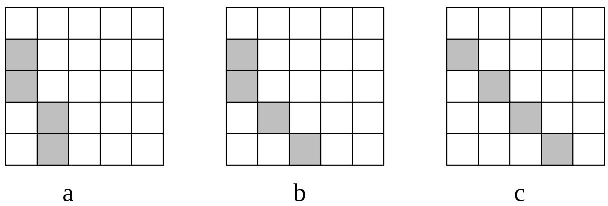

Fig. 6. The three distinct configurations **a-c** for prime cases of type $(5, 5, 4)$

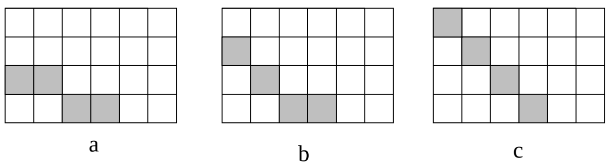

Fig. 7. The three distinct configurations **a-c** for prime cases of type $(4, 6, 4)$

Theorem 9. *There are infinitely many prime configurations.*

Proof. Given a germ of type $(m, m, m^2 - 5m + 4)$, one can construct the following prime configuration: the first point is seen in all images. The remaining $m - 1$ points are seen in exactly 4 images each. Of these $m - 1$ points, the first 4 cameras see exactly 3 points, and the remaining $m - 4$ cameras see exactly 4 points. The construction is illustrated in Fig. 8. For $\widetilde{m} \leq m - 2$: in order to use as much information as possible one should choose the $\widetilde{m}$ cameras close together. This gives in the best case $3\widetilde{m} + 2\widetilde{m} - 4 = 5\widetilde{m} - 4$ unknowns and $\widetilde{m} + 4(\widetilde{m} - 3) + 3 \cdot 2 = 5\widetilde{m} - 6$ constraints, so in this case it is always underdetermined. For $\widetilde{m} = m - 1$ the same reasoning gives at best $3\widetilde{m} + 2(\widetilde{m} + 1) - 4 = 5\widetilde{m} - 2$ unknowns and $\widetilde{m} + 4(\widetilde{m} - 3) + 3 \cdot 3 = 5\widetilde{m} - 3$ constraints. So also in this case it is always underdetermined. Finally, for $\widetilde{m} = m$ we have $3\widetilde{m} + 2\widetilde{m} - 4 = 5\widetilde{m} - 4$ unknowns matching the $\widetilde{m} + 4(\widetilde{m} - 1) = 5\widetilde{m} - 4$ constraints.

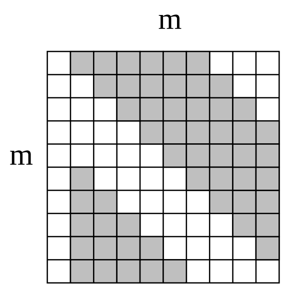

Fig. 8. A prime configuration of type $(10, 10, 54)$

5.8 Extensions

Comparing Table 4 with Table 5, we see that there is only one prime case for four points. Similarly, there is only one prime case for three cameras. The extensions in these cases are of type $(m, 0)$ and $(0, n)$. These types of extensions can always be solved using resection and intersection, respectively. Extensions of type $(1, n)$ and $(m, 1)$ can always also be solved using only combinations of resection and intersection. The first more complicated extension occurs for the type $(2, 2)$. In order for the extension to be unsolvable with intersection and resection all cameras and points must be underdetermined with respect to the prime configuration. And all cameras and points should be exactly determined with the information contained in the remaining four measurements. For an extension of type $(2, 2)$ this can essentially only be done in one way. This extension is shown in Fig. 9. It would be interesting to classify prime ex-

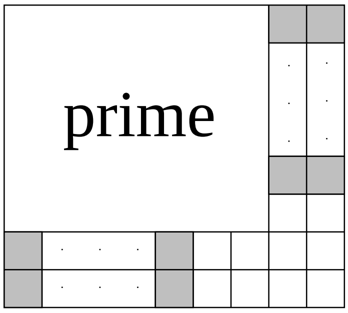

Fig. 9. The extension of type (2,2)

tensions. This would make it possible to express any minimal case as a prime problem extended by a number of prime extensions.

5.9 Classification of 2D Retina Problems

The tools of Sects. 5.1 and 5.5 were developed for calibrated cameras with 1D retina viewing point features. These methods work equally well for other types of cameras and other types of features. In this section we will look at the classification of minimal problems for uncalibrated cameras with 2D retina, where the features are points.

Table 6. The number of excess constraints for m images of n points

	n					
m	6	7	8	9	10	11
2	−1	0	1	2	3	4
3	0	3	6	9	12	
4	1	6	11	16		
5	2	9	16			
6	3	12				
7	4					

A projective camera has 11 degrees of freedom, and a point has 3 degrees of freedom. Each point gives two constraints on the camera. In addition, we have the freedom to choose a projective coordinate system that has 15 degrees of freedom. This means that for m cameras viewing n points the number of excess constraints l is:

$$l = 2mn - 11m - 3n + 15. \tag{40}$$

This number is shown in Table 6. Since an occlusion will reduce the number of constraints by two, it follows that there are minimal cases only when l is even. This means that we only have minimal cases for m and n points when $m+n$ is odd, see Table 6 and Eq. (40).

We can now use the same procedures as we did in Sects. 5.1 and 5.5 for 1D retina for classifying the minimal cases. The number of prime configurations

Table 7. The number of prime configurations for m 2D cameras and n points

	n					
m	6	7	8	9	10	11
2	–	1	–	0	–	0
3	1	–	1	–	1	
4	–	1	–	14		
5	0	–	26			
6	–	4				
7	0					

for 2D retina is shown in Table 7. There is only one prime case for two cameras, the unoccluded case. The same is true for six points.

For the case of three uncalibrated views of 8 points there are 6 distinct germs. Of these 6 germs, 4 are ill-defined, and of the two remaining minimal problems one is prime. This prime problem was solved in [17]. For three views of 9 points there are no minimal configurations. For three views of 10 points there are 33 distinct germs (30 are ill-defined, two are minimal but not prime and one is prime, Fig. 10.) Just like the case of 1D retina, there are infinitely

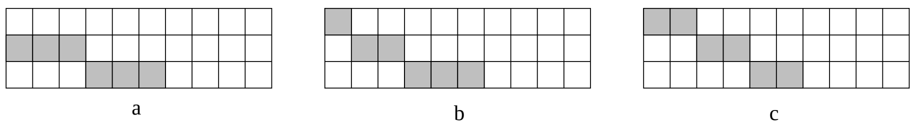

Fig. 10. The three minimal cases of type $(3,10,6)$ for 2D retina. Of these (a) and (b) are minimal but not prime, and (c) is prime

many prime configurations for 2D retina. This can be shown by a similar construction as was done for 1D retina in the proof of Theorem 9. Given m cameras, $m+3$ points and $(2m^2-7m+15)/2$ occlusions, where the first four points are visible in each view, and every subsequent point is visible in three views (Fig. 11) then this is a prime configuration. The proof is completely analogous to the proof of Theorem 9.

5.10 Solution of Some Prime Problems

We now turn our attention to the task of solving some of the prime problems for 1D retina. As such, we will consider the problem with both beacons and

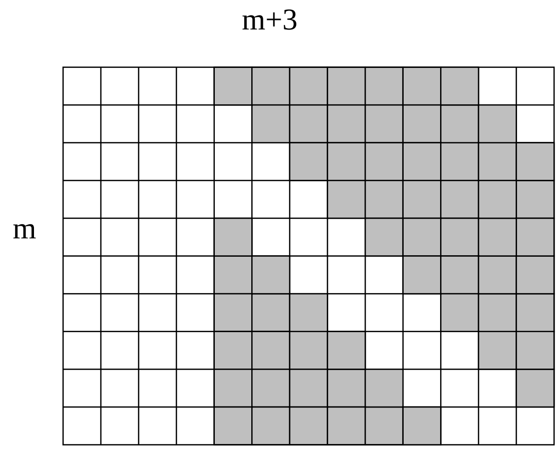

Fig. 11. A prime configuration with m cameras and $m+3$ points. This configuration can be constructed for any m and is always prime

cameras in general positions. As in ordinary vision there exist so-called critical configurations where there is an inherent ambiguity of the solutions to the structure and motion problem irrespective of the number cameras and points. In this chapter we assume noncritical configurations. For nonmissing data and 1D retina the issue of critical configurations was completely resolved in [16]. For missing data it is not known what the critical configurations are, but in order to understand which they are, an understanding of the minimal cases for missing data is desirable.

In the previous section it was also shown that some minimal problems can be solved by extending a prime problem. One such case of a prime extension is also considered in this chapter.

5.11 The Case of Five Points in Four Images

There is only one prime configuration for the case of 5 points in 4 images. This is the case where one sees all 5 points in 2 images. In image 3, one point is occluded and in image 4 another point is occluded. We will start by finding the solutions to this case.

Theorem 10. *The structure and motion problem with 4 views of 5 points*

$$\lambda_{IJ}\mathbf{u}_{IJ} = \mathbf{P}_I\mathbf{U}_J, \qquad \forall (I,J) \in \mathbb{I}\,,$$

with $\mathbb{I}$ such that point 1 is missing in view 3 and point 2 is missing in view 4 (see Table 8) has in general three solutions.

5.12 The Case of Five Points in Five Images

There are three prime problems for the case of 5 points in 5 images. We will now solve these three prime problems and their three dual cases.

Theorem 11. *The structure and motion problem for 5 images of 5 points,*

$$\lambda_{IJ}\mathbf{u}_{IJ} = \mathbf{P}_I \mathbf{U}_J, \qquad \forall (I,J) \in \mathbb{I} \, .$$

with $\mathbb{I}$ *given by Fig. 6a has in general three solutions.*

The dual to this case of 5 points in 5 images is the case of 6 points in 4 images given by Fig. 7a. This means that there are three solutions to this case of 6 points in 4 images.

Corollary 3. *The structure and motion problem for 4 images of 6 points,*

$$\lambda_{IJ}\mathbf{u}_{IJ} = \mathbf{P}_I \mathbf{U}_J, \qquad \forall (I,J) \in \mathbb{I} \, ,$$

with $\mathbb{I}$ *given by Fig. 7a has in general three solutions.*

Using the same kind of parameterization as in the previous cases, we can solve the prime problem given by figure 6b.

Theorem 12. *The structure and motion problem for five images of five points,*

$$\lambda_{IJ}\mathbf{u}_{IJ} = \mathbf{P}_I \mathbf{U}_J, \qquad \forall (I,J) \in \mathbb{I} \, ,$$

with $\mathbb{I}$ *given by Fig. 6b has in general four solutions.*

The dual to this case of 5 points in 5 images is the case of 6 points in 4 images given by Fig. 7b. This means that there are three solutions to this case of 6 points in 4 images.

Corollary 4. *The structure and motion problem for 4 images of 6 points,*

$$\lambda_{IJ}\mathbf{u}_{IJ} = \mathbf{P}_I \mathbf{U}_J, \qquad \forall (I,J) \in \mathbb{I} \, ,$$

with $\mathbb{I}$ *given by Fig. 7b has in general four solutions.*

Finally, the last prime case of 5 images of 5 points can be shown to have five solutions.

Theorem 13. *The structure and motion problem for 5 images of 5 points,*

$$\lambda_{IJ}\mathbf{u}_{IJ} = \mathbf{P}_I \mathbf{U}_J, \qquad \forall (I,J) \in \mathbb{I} \, ,$$

with $\mathbb{I}$ *given by Fig. 6c has in general five solutions.*

The dual to this case of 5 points in 5 images is the case of 6 points in 4 images given by Fig. 7c. This means that there are five solutions to this case of 6 points in 4 images.

Corollary 5. *The structure and motion problem for 4 images of 6 points,*

$$\lambda_{IJ}\mathbf{u}_{IJ} = \mathbf{P}_I \mathbf{U}_J, \qquad \forall (I,J) \in \mathbb{I} \, .$$

with $\mathbb{I}$ *given by Fig. 7c has in general five solutions.*

5.13 The Two-by-Two Extension

Apart from simple extensions based on intersection and resection, the first extension of a prime problem is the extension by two cameras and two points. This type of extension is shown in Fig. 9. We assume that we have a solution to a prime problem with m cameras and n points. The task is then to extend this solution to the solution of the extended problem with $m+2$ cameras and $n+2$ points. Two extra cameras and two extra points means that we have $2 \cdot 3 + 2 \cdot 2 = 10$ unknowns to solve for. Using intersection, the known cameras give two linear constraints on the unknown points. And using resection the known points give four linear constraints on the unknown cameras. This leaves four parameters, one for each camera (A_I, $I = 1, 2$) and one for each point (U_J, $J = 1, 2$), to solve for. The two new points are seen in both the two new views. This gives four quadratic constraints on the four parameters,

$$a_{IJ} U_J A_I + b_{IJ} U_J + c_{IJ} A_I + d_{IJ} = 0; \quad I = 1,2, \ J = 1,2;$$

with the coefficients $(a_{IJ}, b_{IJ}, c_{IJ}, d_{IJ})$ only depending on the images. This means that there could be up to $2^4 = 16$ solutions according to Bezout's theorem, cf. [10]. But because of the sparseness of the polynomials this is not the case. Taking resultants pairwise we can eliminate U_J. This leaves two polynomial equations in A_I:

$$a'_J A_1 A_2 + b'_J A_1 + c'_J A_2 + d'_J = 0, \quad J = 1, 2.$$

Taking the resultant of the two polynomials with respect to A_2 leaves the following quadratic equation in A_1:

$$a'' A_1^2 + b'' A_1 + c'' = 0.$$

This discussion leads to Theorem 14:

Theorem 14. *Given an extension of type* $(2,2)$ *to a structure and motion problem with* m *cameras and* n *points (as depicted in Fig. 9), the number of solutions are in general* $2 \times N$, *where* N *is the number of solutions of the original problem with* m *cameras and* n *points.*

5.14 Some Experimental Results

The methods described in the proof of Theorem 10 can easily be implemented. In Table 8 bearings for an example of the minimal case described in Theorem 10 is shown. The resulting solutions are illustrated in Fig. 12. In this case there were three real solutions with all depths positive.

In Table 9 a setting with 5 cameras and 7 points is shown. This is a minimal problem but not a prime one. The subproblem of the first 3 cameras and the first 5 points is a prime problem which can be solved using algorithm for three views of five points. This gives two solutions. The solution of the whole

Table 8. Some bearing measurements

0.6929	−0.7825	−1.9347	0.3263	−0.6421	
	0.3206	−0.9479	−1.8732	−0.0041	−0.8289
	–	−2.5202	2.4474	−0.9746	−2.3323
	2.3024	–	−1.0540	1.8991	0.6499

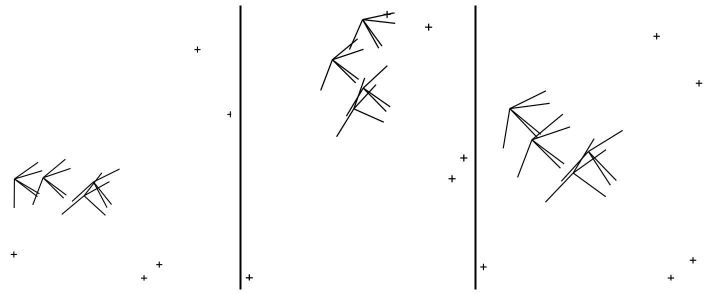

Fig. 12. Three solutions to the minimal case of 5 points in 4 images. Beacons are indicated by '+'

Table 9. Some bearing measurements

−1.9786	−0.5736	0.8046	−1.0507	0.5931	–	–
−2.5202	−1.1710	1.5796	−1.8684	1.2136	–	–
−0.8188	0.6134	3.0339	−0.0885	2.6759	−0.1730	−2.4311
−2.2663	−0.6898	–	–	–	−1.3617	1.5151
−1.8792	−0.3047	–	–	–	−1.1115	2.5170

problem can then be found by the $(2,2)$ extension described in Sect. 5.13. This gives two solutions for each of the original solutions, in total four solutions. In this particular case the solutions all had positive depths. The resulting solutions are shown in Fig. 13.

6 Ambiguous Cases of Structure and Motion Problems

We have seen previously that a solution to the structure and motion problem is only determined up to an unknown projective transformation. Also, for three cameras and any number of points, there is accordingly a twofold ambiguity. Additionally, there are two basic ambiguities that will be discussed here.

For the intersection problem there is one critical configuration for which there is not a unique solution.

Theorem 15. *Consider the case of several views of one point with known camera matrices. The intersection problem is ambiguous if and only if all camera centres and the point lie on a line.*

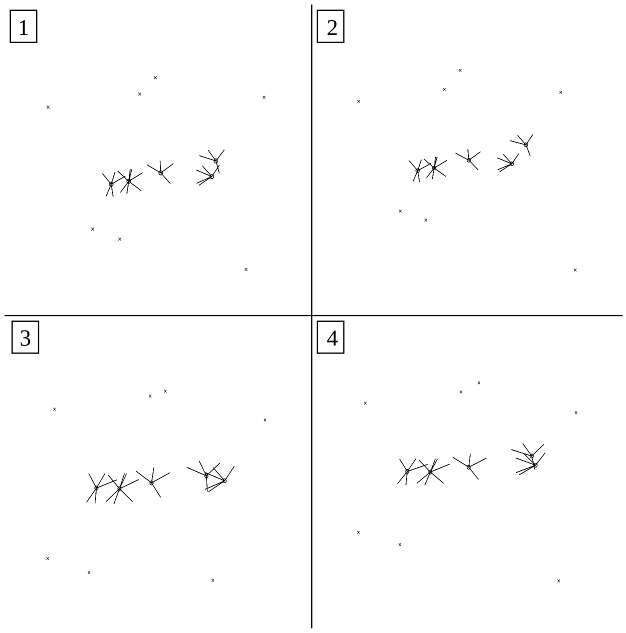

Fig. 13. Four solutions to one minimal case of 7 points in 5 images. Beacons are indicated by '+'

Resection is the problem of calculating camera positions using image measurements and known object points. In this case, the critical configurations are not obvious.

Theorem 16. *Consider $m > 4$ object points with known positions and one unknown camera. The resection problem is ambiguous if and only if all points and the camera centre lie on a conic curve.*

Proof. Consider first the critical configuration for the intersection problem, i.e. 1 point and $n > 1$ camera centres lying on a single line. The configuration is still ambiguous if more object points are added. Four points (where no three are on a line) and several $n > 1$ cameras are ambiguous if and only if 1 of the points and all camera centres are on a single line. If the other 3 points are taken as base points, the dual statement is: $m > 4$ object points and one camera centre are ambiguous if and only if all points and the camera centre lie on a conic curve.

The intersection and resection ambiguities are illustrated in Fig. 14. The calibrated version follows.

Corollary 6. *Consider $m > 2$ object points with known positions and one unknown camera. The calibrated resection problem is ambiguous if and only if all points and the camera centre lie on a circle.*

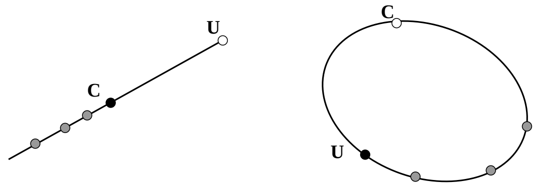

Fig. 14. (left) The intersection ambiguity where a point **U** and several camera centres **C** lie on a line. **(right)** The dual resection ambiguity where a camera centre **C** and several points **U** lie on a conic curve

6.1 Three View Ambiguities

A structure and motion problem with three views can be ambiguous in three ways

1. The alternative reconstructions have the same relative camera motion.
2. The alternative reconstruction have different relative camera motion, but the corresponding trilinear tensor is the same.
3. The alternative reconstructions have different relative camera motion, and the corresponding trilinear tensor is different.

For case 1 there is a unique relative motion, so one can without loss of generality assume that the camera positions are known. The alternative reconstruction differs in at least one of the object points. This can only happen if the camera centres and that point are collinear, see Theorem 15. For case 2, Theorem 2 shows that for each trilinear tensor there are two possible relative orientations. Thus any three-view problem is critical in the sense that there are at least two possible solutions. For the third case, we ask if there are cases where there might be more than two solutions to the structure and motion problem, i.e. when the tensor is not uniquely defined. We will call this case a *three-view ambiguity.*

We are now ready to state the theorem describing exactly when there are three-view ambiguities. For an example, see Fig. 15.

Theorem 17. *The structure and motion problem for three views and arbitrary number of points is ambiguous if and only if the three camera centres and all the object points lie on a cubic curve.*

There is an interesting special case when all the points and at least one of the camera centres lie on a conic. It fits into the theorem since there is a cubic consisting of the conic through the points and one camera centre and a line through the remaining camera centres. The cubic thus covers all points and camera centres. The problem is then critical in the sense that the resection problem for the first camera is critical, cf. Theorem 16.

Proof. Consider a situation where there is an ambiguity. Consider one of the solutions to the problem. For this solution there is a placement of cameras,

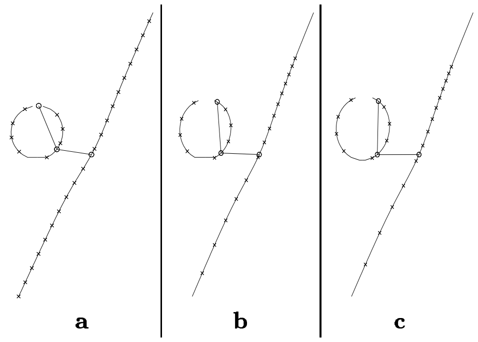

Fig. 15. Three cameras (*circles*) are viewing 22 points (*crosses*). All three configurations **(a–c)**(out of a one-parameter family) are consistent with the 1D image points. The 25 plane points lie on a cubic

A, **B** and **C**. The condition that there is an ambiguous solution is equivalent to saying that there is an alternative tensor T_{ijk} such that

$$\sum T_{ijk}\mathbf{a}^i\mathbf{b}^j\mathbf{c}^k = 0\,,$$

where **a**, **b** and **c** are image points in the three images, respectively. Since

$$\mathbf{a}^i = \mathbf{A}^i\mathbf{X}, \quad \mathbf{b}^i = \mathbf{B}^i\mathbf{X}, \quad \mathbf{c}^i = \mathbf{C}^i\mathbf{X},$$

the constraint on the object point is a third-degree polynomial in $\mathbf{X} \in \mathbb{P}^2$:

$$p(\mathbf{X}) = \sum T_{ijk}\mathbf{A}^i\mathbf{X}\mathbf{B}^i\mathbf{X}\mathbf{C}^i\mathbf{X} = 0\,.$$

This shows that all object points pass through this cubic curve. To see that the camera centres lie on the same curve it is sufficient to observe that $\mathbf{AF} = 0$, when **F** is the camera centre for camera 1. This gives directly that $p(\mathbf{F}) = 0$. Notice that the structure and motion problem is in general well defined for 7 points in 3 views. With 6 object points and 3 camera centres, there is in general a unique cubic curve passing through these points. That the 7th object point also lie on this curve is exceptional. To show the only if part, we consider an object where all camera centres and object points lie on an arbitrary third-degree polynomial. Without loss of generality we may change both object coordinate system and image coordinate system so that

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\,,$$

as long as the three cameras are not on a line. The mapping from ambiguous tensors to cubic curves is a linear mapping. Each ambiguous tensor that has eight parameters $T = (T_{111}, T_{112}, T_{121}, T_{122}, T_{211}, T_{212}, T_{221}, T_{222})$ corresponds to a cubic curve

$$p(\mathbf{X}) = \sum T_{ijk} \mathbf{A}^i \mathbf{X} \mathbf{B}^i \mathbf{X} \mathbf{C}^i \mathbf{X} = 0 \ ,$$

where the coefficients $c = (c_{x^3}, c_{x^2y}, c_{x^2z}, c_{xy^2}, c_{xyz}, c_{xz^2}, c_{y^3}, c_{y^2z}, c_{yz^2}, c_{z^3})$ of the polynomial $p(\mathbf{X})$ depend linearly on the tensor coefficients

$$c = MT \, . \tag{41}$$

For this particular choice of coordinates the matrix M becomes

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \, .$$

It is straightforward to see that the matrix M has rank 7. Notice that the true tensor is a null vector to the matrix, so M must have rank ≤ 7. If the three camera centres happen to be on a line, it is easy to check that the corresponding mapping is also linear with rank 7. The mapping given by Eq. (41) is in fact a bijective mapping from the star of tensors through the true tensor (which can be identified with $\mathbb{P}^6$) to the manifold of cubic curves that pass through the three camera centres (also $\mathbb{P}^6$).

Since the mapping is bijective, our arbitrary third-degree curve on which the object points lie corresponds to an ambiguous tensor. Thus the structure and motion problem for that case is critical. This concludes the proof.

From the principle of duality, the following theorem is obtained.

Theorem 18. *The structure and motion problem for any number of views of 6 points is ambiguous if and only if the camera centres and the object points lie on a cubic curve.*

Proof. The image under the Carlsson map of a cubic curve through the base points is again a cubic curve through the base point. The dual of 3 cameras and n points is $m = n - 3$ cameras and 6 points. So by the principle of duality and Theorem 17, the statement is proved.

6.2 General n Points in m Views Ambiguity

Up to now, we have limited either the number of cameras or the number of points considered. Based on the previous results, the general problem will now be solved. A natural generalization of the three-view case for the word "ambiguous" is that the alternative reconstructions have different relative camera motion, and (at least) one triplet of cameras has a different trilinear tensor.

Theorem 19. *A 1D structure and motion problem is ambiguous regardless of the number of cameras and points if and only if all the camera centres and the object points lie on a common third-degree curve.*

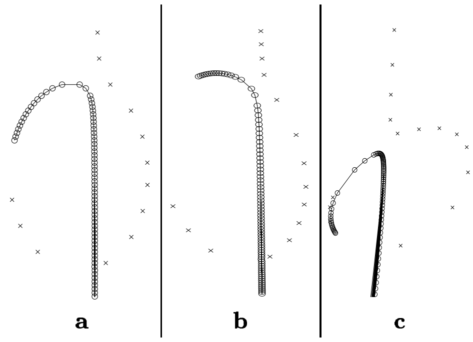

Fig. 16. The figure illustrates three solutions **(a–c)** (out of a one-parameter family) to the same structure and motion problem. There are 82 cameras (*circles*) viewing 15 points (*crosses*). It is critical because all 97 points lie on a cubic

Proof. We begin by showing that a problem is ambiguous if all points lie on a third-degree curve. Assume that camera centres and object points lie on a third-degree curve c. By first restricting the problem to only 6 points, we know from Theorem 18 that the configuration is ambiguous and there is (at least) a one-parameter family of solutions. Now, consider a seventh point on the curve c. We need to show that the constraints generated by the projection equation for this extra point do not break the ambiguity. However, all these constraints reduce to trilinear constraints, as there are no higher-order constraints for 1D camera motion [6]. Thus, it suffices to consider three arbitrary cameras $\mathbf{P}_i$, $\mathbf{P}_j$ and $\mathbf{P}_k$. In the proof of Theorem 17, we showed that the map from stars of tensors to cubic curves (through the camera centres) is bijective. So, from c and the three camera centres, a star of tensors $\lambda T_1 + \mu T_2$, where $(\lambda, \mu) \in \mathbb{P}^1$, is obtained. But according to the proof of Theorem 17, as long as the seventh point is on c, all tensors in $\lambda T_1 + \mu T_2$ are still valid solutions.

To show that each ambiguous problem has the property that all points lie on a third-degree curve we use a proof by contradiction. Thus, assume that there exist ambiguous problems with m views of n points such that the $m+n$ points do not lie on a common third-degree curve. Such problems must have $m > 3$ and $n > 6$ because of Theorems 17 and 18, respectively. Study such a problem where $m + n$ is minimal. If we remove one point or one camera, we obtain an ambiguous problem with one point less. By the assumption, these $m + n - 1$ points must lie on a third-degree curve. In particular, this means that all 10 point subconfigurations must lie on a third-degree curve. But then all $m + n$ points lie on a cubic curve. Thus all ambiguous configurations have the property that all $m + n$ points lie on a third-degree curve.

In Fig. 16 an example of a critical configuration is illustrated. Even though there are 82 views of 15 points, the 1D images alone cannot disambiguate between a one-parameter family of solutions. Another example is illustrated

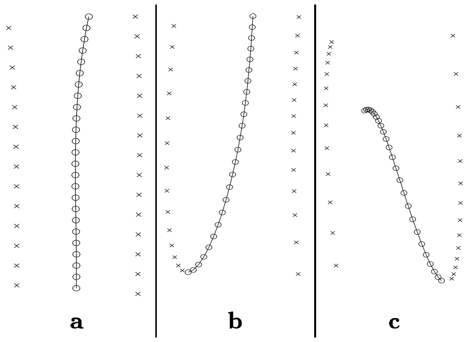

Fig. 17. Three solutions (out of a one-parameter family) to the same structure and motion problem. In the example, the camera moves in a corridor with scene points on both walls, which is quite common in robot navigation. There are 25 cameras (*circles*) viewing 29 points (*crosses*). It is critical because all 54 points lie on a cubic.

in Fig. 17, where a camera moves along a corridor, which frequently occurs in practical situations.

7 Conclusions

In this paper we introduced the minimal conditions for solving the structure and motion problem for cameras with one-dimensional retinae. The emphasis was on calibrated cameras.

For the minimal case of 3 images with 5 points it was shown how to solve the problem using the calibrated trilinear tensor. It was shown that there is a two-to-one map from the relative orientation of three cameras to the calibrated trilinear tensor. This explains why there are two solutions to the structure and motion problem for three cameras. A geometric interpretation of this ambiguity is also given.

For the minimal case of 4 images with 4 points it was shown how to solve the problem using the dual calibrated quadrilinear tensor. It was shown that there is a two-to-one map from the shape of four planar points to this tensor. This explains why there are two solutions to the structure and motion problem for four points.

Notice that the trilinear tensor encodes the relative motion of the camera at three instants and that the dual quadrilinear tensor encodes the structure of four scene points. Thuss there are no relations at all between these two tensors.

The connection between the calibrated and the uncalibrated cameras is given. From this it follows that similar results hold for 3 images with 7 points and for 4 images with 6 points. Using the Carlsson duality it was then shown that the above two types of ambiguities are in fact dual to each other.

Furthermore, a categorisation of minimal cases for structure and motion is given. Some of the simpler minimal cases are solved. It is shown that there are infinitely many such minimal cases.

Finally a complete categorisation of ll ambiguous configurations for the structure and motion problem in 1D retina vision is presented. The main ambiguity is when all object points (regardless of how many) and all camera centres (again, regardless of the number of cameras) lie on a cubic curve.

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