Fundamental Limitations on Projective Invariants of Planar Curves

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***Abstract*— In this paper, some fundamental limitations of projective invariants of non-algebraic planar curves are discussed. It is shown that all curves within a large class can be mapped arbitrarily close to a circle by projective transformations. It is also shown that arbitrarily close to each of a finite number of closed planar curves there is one member of a set of projectively equivalent curves. Thus a continuous projective invariant on closed curves is constant. This also limits the possibility of finding so called projective normalisation schemes for closed planar curves.**

***Keywords*— Projective and affine invariants, recognition, Hausdorff metric.**

I. Introduction

The pinhole camera is often an adequate model for projecting points in three dimensions onto a plane. Using this model it is straightforward to predict the image of a collection of objects in specified positions. The inverse problems, to identify and to determine the three-dimensional positions of possible objects from an image, are however much more difficult. Traditionally recognition has been done by matching each model in a model data base with parts of the image. Recently, model based recognition using viewpoint invariant features of planar curves and point configurations has attracted much attention, cf. [7]. Invariant features are computed directly from the image and used as indices in a model data base. This gives algorithms which are significantly faster than the traditional methods. These techniques cannot, however, be used to recognise general curves or point features in three dimensions by means of one single image. Additional information, e.g. that the object is planar, is needed. For point configurations the reason is that only trivial invariants exist in the general case, as is shown in [4,9]. In this paper it is shown that there are some fundamental limitations also for planar curves.

More specifically, two theorems are presented that elucidate these limitations. The first one, in Section II, states that each curve in a large class can be transformed into a curve arbitrarily close to a circle in a strengthened Hausdorff metric. The second theorem, in Section III, states that given a finite number of closed planar curves $\Gamma_1, \ldots, \Gamma_m$, it is possible to construct a set of *projectively equivalent* planar curves $\Gamma'_1, \ldots, \Gamma'_m$, such that Γ'_i in the Hausdorff metric is arbitrarily close to Γ_i, $i = 1, \ldots, m$. These two theorems enlighten the limitations of invariant based recognition schemes. The first one tells us that choosing a distinguished frame by maximising some feature over all projective transformations is not suitable, since in the limit many curves look like circles. The second theorem tells us more generally that every continuous invariant must be constant. Some consequences of these theorems will be discussed in Section IV. Their relevance to computer vision is that the euclidean errors in image processing do not interact well with projective equivalence.

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II. Many curves look like a circle

This paper is concerned with the shape of curves and the effect of projective transformations on curves. According to the pinhole camera model such transformations appear naturally in the study of vision systems. The following notations will be used.

Let $\mathcal{C}$ be the set of all curves which can be represented as a continuous injective mapping from the unit circle to the plane, such that the arclength l is well defined. Let $A(C)$ denote area enclosed by the curve C. It is a well known fact from the calculus of variations that $l(C)^2/A(C) \geq 4\pi$, with equality if and only if C is a circle. For a specific curve $C \in \mathcal{C}$, let P_C be the set of projective transformations that sends C into $\mathcal{C}$. In other words such transformations do not send any of the points of C to infinity. Two images of the same planar curve, caught by a pinhole camera, are always related by such a transformation.

A metric on $\mathcal{C}$ is defined by

$$d(C_1, C_2) = \max_{z_1 \in C_1} \min_{z_2 \in C_2} ||z_1 - z_2|| + \max_{z_2 \in C_2} \min_{z_1 \in C_1} ||z_1 - z_2|| + |l(C_1) - l(C_2)|. \quad (1)$$

Here $||x||$ is the euclidean norm. This metric is a modification of the Hausdorff metric on compact subsets of the plane, i.e. the max-min parts, the modification being that also the arclengths should be compared. A small value of d depicts that every point on each curve is close to some point on the other curve, and that the arclengths are almost equal. This metric will be used to compare two projected curves in the image plane. Due to digitisation effects and other errors in image acquisition, it is difficult to discriminate two image curves that are close in this metric. These errors are euclidean by nature. Theorems 1 and 3 below are automatically valid also in the ordinary Hausdorff metric. The modified metric is needed for the proof of Corollary 2.

Let $\widetilde{\mathcal{C}} \subset \mathcal{C}$ consist of those curves in $\mathcal{C}$ having the property that the boundary of the convex hull has at least one smooth, curved part.

Theorem 1 *Let C_0 be a circle of radius one. Then*

$$C \in \widetilde{\mathcal{C}} \Longrightarrow \inf_{p \in P_C} d(p(C), C_0) = 0.$$

One interpretation of this theorem is that for some sequence of viewpoints and internal calibration the images of C look more and more like a circle. As will be seen in the proof below the projective transformations involved when approaching the limit are quite extreme, but still non-singular.

Proof:

Choose a point $a \in C$ so that C is smooth at a, and so that the tangent at a intersects C only at a. Choose a coordinate system with origin at a, with x-axis along the tangent, and so that the curvature at a equals one.

The idea of the proof is to construct a sequence of transformations $(p_n)_1^\infty$ so that $p_n(C) \longrightarrow C_0$ as $n \longrightarrow \infty$, in the metric d. The image of a part of the curve around a will form the main part of C_0, and the remaining part of C will be mapped into a neighbourhood of one particular point of C_0.

The transformations p_n are defined by

$$p_n(x,y) = \left(\frac{2nx}{(n^2-1)y+2}, \frac{2n^2 y}{(n^2-1)y+2} \right). \qquad (2)$$

We will also use the ellipses

$$C_\epsilon = \{((1+\epsilon)\cos t, \sin t + 1) | t \in \mathcal{R}\}, \quad \epsilon > -1 \qquad (3)$$

with center at the point $(0,1)$, axis of length $1+\epsilon$ in the x-direction and of length 1 in the y-direction. In particular, C_0 is the unit circle $x^2 + (y-1)^2 = 1$. These ellipses intersect at $(0,0)$ and at $(0,2)$.

One can easily verify, e.g. using homogeneous coordinates, that the family $(p_n)_1^\infty$ has the following properties:

$$\begin{aligned} p_1 &= \text{identity} && (4)\\ p_a \circ p_b &= p_{ab} && (5)\\ p_n(C_\epsilon) &= C_\epsilon && (6)\\ p_n(0,0) &= (0,0) && (7)\\ p_n(0,2) &= (0,2) && (8) \end{aligned}$$

By (6), the transformations p_n reparametrise the ellipses C_ϵ. It will be seen that if $n > 1$ a vicinity around $(0,0)$ expands and a vicinity around $(0,2)$ contracts. More precisely, by rewriting (2) as

$$p_n(x,y) = \left(\frac{2nx}{(n^2-1)y+2}, 2 + \frac{2y-4}{(n^2-1)y+2} \right), \qquad (9)$$

it follows that for every compact region D in the open upper half plane $\{(x,y) | \quad y > 0\}$,

$$\sup_D |p_n(x,y) - (0,2)| \leq K/n, \qquad (10)$$

for some constant K. Hence $(p_n)_1^\infty$ is uniformly convergent to the constant function $(0,2)$ on D. Since the Jacobians of p_n are uniformly bounded by $O(1/n)$ on D, it also follows that the transformations p_n are uniformly Lipschitz continuous with Lipschitz constant $O(1/n)$ on D, i.e.

$$|p_n(x_1,y_1) - p_n(x_2,y_2)| \leq K/n |(x_1,y_1) - (x_2,y_2)|,$$
$$\forall (x_1,y_1) \in D, \forall (x_2,y_2) \in D, \forall n \ . \qquad (11)$$

By changing coordinates with $t(x,y) = (-x, 2-y)$ the inverse projective transformation p_n^{-1} is equal to p_n, i.e. $p_n^{-1} = t^{-1} \circ p_n \circ t$. Thus the inverse transformations p_n^{-1} also have the contractive properties (10) and (11) in every compact region D in the open half plane $\{(x,y) | \quad y < 2\}$. All points of $p_n^{-1}(D)$ tend to $(0,0)$ and the arclength of all curves tend to zero as n increases.

Notice again that as n increases so does both the contractive properties of p_n on every compact region above the tangent to C_0 at $(0,0)$ and the contractive properties of p_n^{-1} on every compact region below the tangent to C_0 at $(0,2)$. This will be used in the proof of Theorem 3.

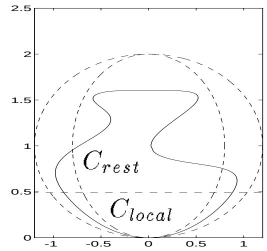


Fig. 1 The curve is split into two parts. A local part C_{local} belongs to the region bounded by the line and the two ellipses. C_{rest} is the complementary part of C.

Take $\epsilon > 0$, and let C_{local} be the connected component of C in a neighbourhood of $(0,0)$, that lies between the ellipses C_ϵ and $C_{-\epsilon}$, cf. Figure 1. Since the curve $p_n(C_{local})$ lies between the ellipses, and these are invariant under p_n, the following inequalities hold,

$$1 - \epsilon < |(u,v) - (0,1)| < 1 + \epsilon, \quad \forall (u,v) \in p_n(C_{local}), \forall n.$$

The rest of the curve, $C_{rest} = C \backslash C_{local}$, is compact and belongs to the upper half plane. By the uniform convergence (10), for each $\epsilon > 0$ we can choose n so that all points of $p_n(C_{rest})$ lie within the distance ϵ from $(0,2)$, cf. Figure 1. Hence

$$\lim_{n \to \infty} (\max_{z_1 \in p_n(C)} \min_{z_2 \in C_0} ||z_1 - z_2|| +$$
$$\max_{z_1 \in C_0} \min_{z_2 \in p_n(C)} ||z_1 - z_2||) = 0. \qquad (12)$$

By this, one has control of the first two terms in the definition of d. A consequence that will be used below, is that $\lim_{n \to \infty} A(p_n(C)) = \pi$.

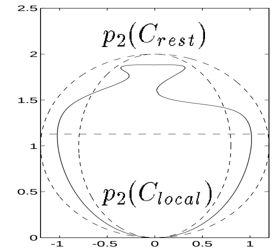
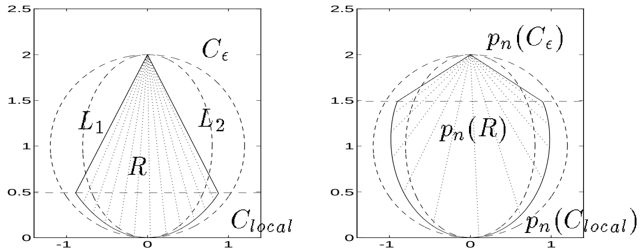


Fig. 2 The local part C_{local} together with two line segments form the boundary of a convex region R. For every n the transformed region $p_n(R)$ is convex and belongs to the interior of the ellipse C_ϵ.

It remains to consider the third term in d. The curve C is smooth around $(0,0)$, so it is possible to choose C_{local} so



small that together with the lines L_1 and L_2 from the endpoints of C_{local} to $(0,2)$, it forms the boundary of a convex region R, cf. Figure 2. Since the shortest path circumventing a bounded region is the boundary of its convex hull, and since $p_n(C_{local})$ is part of the boundary of the convex region $p_n(R)$, we can deduce that $l(p_n(C_{local})) < l(C_\epsilon)$ for all n. By comparison with a circle of radius $1+\epsilon$ we get $l(C_\epsilon) < 2\pi(1+\epsilon)$. Since C_{rest} lies in a compact subset of the open upper half plane, by means of (11) we have

$$\limsup_{n\to\infty} l(p_n(C)) \le \limsup_{n\to\infty} l(p_n(C_{local})) + \limsup_{n\to\infty} l(p_n(C_{rest})) \le 2\pi(1+\epsilon)+0.$$

Hence $\limsup_{n\to\infty} l(p_n(C)) \le 2\pi$. On the other hand, since $l(p_n(C))^2/A(p_n(C)) \ge 4\pi$, it follows that

$$\liminf_{n\to\infty} l(p_n(C)) \ge 2\pi.$$

Hence $\lim_{n\to\infty} l(p_n(C)) = l(C_0) = 2\pi$, which concludes the proof.

■

An immediate corollary is

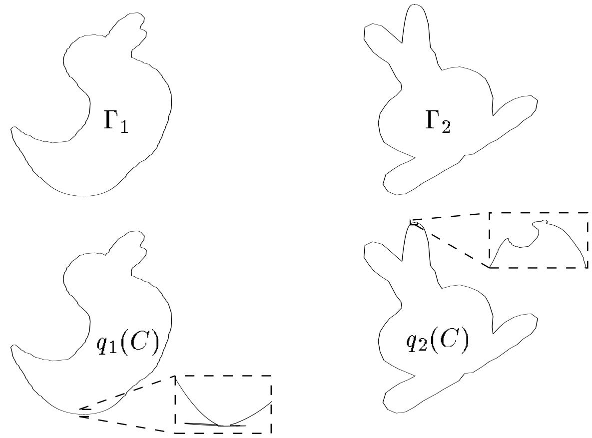
Corollary 2

$$C \in \widetilde{\mathcal{C}} \Longrightarrow \inf_{p\in P_C} \frac{l(p(C))^2}{A(p(C))} = 4\pi$$

It has been proposed, e.g. in [2], to base a canonical representation $\bar{p}(C)$ of the curve C on the transformation $\bar{p}$ that minimises the inverse compactness measure $l(p(C))^2/A(p(C))$. According to the corollary, the minimum is not attained if $C \in \widetilde{\mathcal{C}}$. This canonical representation is thus only well defined for curves that do not have a smooth and curved part on the convex hull, e.g. for polygons. However, it is still possible that local minima could be used, even for curves in $\widetilde{\mathcal{C}}$.

III. Projecting a duck to a rabbit

In the proof of Theorem 1 one notices that the main part of the curve is squeezed into a neighbourhood of a point. For large n, the curve $p_n(C)$ looks like a circle, but has a small ripple that corresponds to the main part of the curve C. It turns out that if we slightly perturb the curve $p_n(C)$ outside this ripple and then do the inverse projective transformation, the new curve is almost identical to the original one. A consequence is the following somewhat surprising theorem.



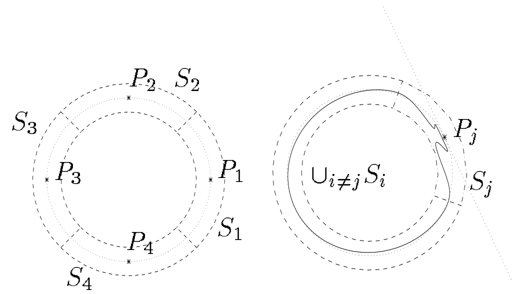


Fig. 3 The upper two curves are not projectively equivalent, but the lower two ones are. The lower curves are constructed by introducing small ripples along the convex hull, these are illustrated in the magnified pictures.

Theorem 3 *Given $\Gamma_1,\dots,\Gamma_m \in \mathcal{C}$. To every $\epsilon > 0$, there exists a curve C and projective transformations $q_1,\dots,q_m$ so that*

$$d(q_i(C),\Gamma_i) < \epsilon, \qquad i = 1,\dots,m.$$

The theorem is illustrated in Figure 3. Notice that the curves Γ_i do not have to be smooth.

Proof: Since there is a smooth curve arbitrarily close to every curve fulfilling the assumptions above, it is no restriction to assume that the curves $\Gamma_1,\dots,\Gamma_m$ are smooth and therefore in $\widetilde{\mathcal{C}}$.

Place m points $(P_j)_1^m$ and m closed regions $(S_j)_1^m$ equally spaced around the unit circle C_0 according to Figure 4. The regions S_j are supposed to form a band around C_0, so thin that $\cup_{i\neq j} S_i$ is disjoint from the tangent to C_0 at P_j.

Fig. 4 The left figure illustrates how the points $(P_j)_1^m$ and the closed regions $(S_j)_1^m$ are placed around C_0 in the case $m = 4$. The curve Γ_j is projected into an almost circular curve $\pi_j(\Gamma_j)$ with a small ripple around P_j. This is illustrated in the right figure.

In the proof of Theorem 1 it was seen that every smooth curve can be projected arbitrarily close to C_0. The main part of the curve forms a small ripple close to a point on the unit circle. In this way not only is it possible to make the whole curve closer to C_0 and the ripple around P_j smaller but at the same time the contractive properties of the inverse transformation on a region like $\cup_{i \neq j} S_i$ is increased, cf. the discussion after (11). By the construction in the proof of Theorem 1 it is thus possible to select a transformation π_j, and also to cut each curve Γ_j into two pieces $\Gamma_{j,local}$, and $\Gamma_{j,rest}$, so that the following properties are obtained:

- $\pi_j(\Gamma_{j,local}) \subset \cup_{i \neq j} S_i$
- $\pi_j(\Gamma_{j,rest}) \subset S_j$
- $d(\pi_j(\Gamma_j), C_0) < 1/m$.
- $l(\pi_j(\Gamma_{j,rest})) < 3\pi/m$.
- $q_j = \pi_j^{-1}$ shrinks all curves in $\cup_{i \neq j} S_i$ of arclength less than a constant $M = 3\pi + 2$ into a curve with arclength less than $\epsilon/2$. The reason for the choice of constant will become clear later.

Let C be constructed by gluing the patches $\pi_j(\Gamma_{j,rest})$ and the line segments obtained by radially connecting the endpoints of $\pi_j(\Gamma_{j,rest})$. Both $C \setminus \pi_j(\Gamma_{j,rest})$ and $\pi_j(\Gamma_{j,local})$ are in $\cup_{i \neq j} S_i$. Since C is a patch of m curves each with arclength less than $3\pi/m$, and of m radial line segments of length less than $2/m$, the total arclength of $C \setminus \pi_j(\Gamma_{j,rest})$ is certainly less than $M = 3\pi + 2$. By the contractive properties of q_j, this means that $l(q_j(C \setminus \pi_j(\Gamma_{j,rest}))) < \epsilon/2$. The curve $\pi_j(\Gamma_{j,local})$ also has arclength less than M, so $l(\Gamma_{j,local})) < \epsilon/2$. Since these curves have the same endpoints, it is then clear that

$$d(q_j(C \setminus \pi_j(\Gamma_{j,rest})), \Gamma_{j,local}) < \epsilon.$$

The remaining part of C is $\pi_j(\Gamma_{j,rest})$, which is mapped identically into $\Gamma_{j,rest}$ by q_j. Hence $d(q_j(C), \Gamma_j) < \epsilon$. ■

Notice that the transformations q_j are physically realisable in the pinhole camera model. The construction of C and q_i in the proof can be done by explicit formulas. An algorithm based on the proof has been implemented in MATLAB. Figure 3 has been constructed using this algorithm. Figure 5 shows what the mixed curve C looks like from eight different viewpoints. Observe that these eight different views are all projectively equivalent. Notice the kind of extreme, but non-singular, projective transformations that are involved.

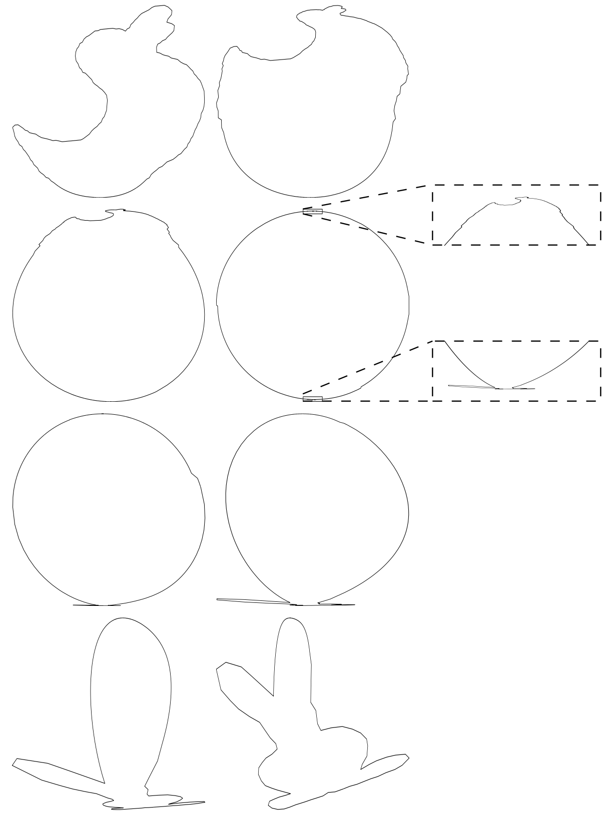


Fig. 5 Eight projectively equivalent views of the same planar curve. The duck transforms into something that looks like a circle and then into a rabbit. A closer look at the fourth curve reveals that the north and south pole is slightly rippled, see the magnifications.

IV. Implications for Invariants

By an invariant under a set of transformations P on $\mathcal{C}$ is meant a function ϕ on $\mathcal{C}$ with values in some set V such that $\phi(C) = \phi(p(C))$ for every curve $C \in \mathcal{C}$ and every transformation $p \in P$. If $\mathcal{C}$ and V are metric spaces, we can talk about continuity of invariants.

One consequence of Theorem 1 is that in every neighbourhood of the circle $N_{\epsilon, C_0} = \{C | d(C, C_0) < \epsilon\}$, ϕ attains every value that it attains on $\widetilde{\mathcal{C}}$. In particular if ϕ is nonconstant on $\widetilde{\mathcal{C}}$, this means that ϕ is discontinuous at C_0.

This is however not a very useful observation. Discontinuities of this kind appear for many of the most useful invariants. For instance whenever the group of transformations contains the similarity group, each object can be contracted into an ϵ-neighbourhood of the origin, where thus ϕ attains all its values and becomes discontinuous. Thus e.g. even the crossratio has discontinuities in this sense, which tells us that the property of having a discontinuity at one point is not very informative.

More interesting conclusions about invariants can be obtained from Theorem 3.

Corollary 4 *Every invariant ϕ from $\mathcal{C}$ to a metric space V, e.g. the real line, maps all curves at which it is continuous onto the same value.*

Proof: Assume to the contrary that $r_1 = T(\Gamma_1) \neq r_2 = T(\Gamma_2)$, and that ϕ is continuous both at Γ_1 and Γ_2. It is possible to find disjoint open sets $O_1 \ni r_1$ and $O_2 \ni r_2$. According to Theorem 3 the inverse images $T^{-1}(O_1)$ and $T^{-1}(O_2)$, which are open sets around Γ_1 and Γ_2, contain a projectively equivalent pair of curves, contradicting the assumption. ■

V. Conclusions

The corollary tells that for invariants the properties of being continuous and discriminating are contradictory. Notice that the theorem only holds if we consider the whole set $\mathcal{C}$. If more information about the curves are given, e.g. if fiducial points are given, then it might be possible to construct invariants which are non-constant and continuous at many places.

Thus the euclidean nature of image distorsion and the projective nature of camera geometry do not interact well.

It is possible that one could construct projective invariants which are continuous with respect to some other metric, but would this other metric be relevant?

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