# Multilinear Constraints in the Infinitesimal-time Case

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**Abstract**

*In this paper we study the infinitesimal-time case of the so called multilinear constraints that exist for each subsequence in a sequence of images. These constraints link the infinitesimal motion of the image points with the infinitesimal viewer motion. The analysis is done both for calibrated and uncalibrated cameras. Two simplifications are also presented for the uncalibrated camera case. One simplification is made using affine reduction and kinetic depth. The second simplification is based upon a projective reduction with respect to the image of a planar patch.*

## 1 Introduction

A central problem in scene analysis is the analysis of 3D-objects from 2D-images, obtained by projections. In this paper we will concentrate on the case of a sequence of images consisting of rigid point configurations, with known correspondences. The objective is to calculate the shape of the object using the shapes of the images and to calculate the camera motion, represented as matrices. The analysis of the uncalibrated camera case makes it possible to reconstruct the object and the camera movement up to a projective transformation, whereas in the calibrated camera structure and motion is obtained up to a similarity transformation.

One interesting question is to analyse the multilinear constraints that exist between corresponding points in an image sequence. It is well known that corresponding points in two images fulfill a bilinear constraint, known as the epipolar constraint. This can be represented by a three by three matrix called the essential matrix in the calibrated case and the fundamental matrix in the uncalibrated case. In the continuous time case similar constraints exist. One talks about the infinitesimal epipole or the focus of expansion and the analysis of optical flow. This has been studied by photogrametrists in the calibrated case and recently by Faugeras and Vieville in the uncalibrated case, cf. [13].

In this paper we derive these multilinear constraints, for both calibrated and uncalibrated cameras, in both discrete and continuous time, and formulate these in what we believe to be a very simple and clear form. This simple formulation has some nice advantages. From the formulation it is apparent that all multilinear constraints can be derived from the bi- and trilinear constraints, see [6, 2, 8]. Similarly it will be shown that in the infinitesimal-time case, the motion is observable from the second order constraint if the camera velocity is non-zero. The difference and the similarities between uncalibrated and calibrated case is clearer. The parameters in the multilinear constraints are closely linked to the camera parameters describing the projection of points in 3D onto each image plane.

## 2 Camera Geometry

In the discussions that follow we will use the pinhole camera model and will use projective and oriented projective geometry or spherical geometry. This is described in detail in [11].

A point in three dimensions will be represented either in the three-dimensional euclidean space $\mathbf{U} = (U_x, U_y, U_z)$ or as a point in the three-dimensional oriented projective space using homogeneous coordinates $U = (U_x, U_y, U_z, 1)$. In oriented projective geometry two representations are considered as the same point if one is a positive multiple of the other.

We will consider a couple of different perspective camera models. Projection unto the image oriented projective image plane is conveniently represented in the camera matrix formulation

$$\lambda u = PU, \qquad \lambda > 0, \tag{1}$$

where $\lambda$ is the unknown depth, and $u$ is the image position, also in oriented homogeneous coordinates $u = (u_x, u_y, 1)$, possibly corrected for the internal calibration if this is known. We will consider different settings, depending on what camera model is used,

$$P = [I| - T], \text{ known rotation and internal calibration,} \tag{2}$$
$$P = R[I| - T], \text{ known internal calibration,} \tag{3}$$
$$P = AR[I| - T], \text{ uncalibrated camera,} \tag{4}$$

where $T = (T_x, T_y, T_z)^T$ is the unknown position of the camera focus, $R$ is a rotation matrix describing the orientation of the camera and $A$ is the matrix representing the unknown internal calibration parameters. In the uncalibrated case when the internal calibration matrix $A$ is unknown it is convenient to let $Q^{-1} = AR$, represent the generalised orientation of the camera. In other words we think of the camera to have a position $T$ and a generalised orientation $Q$. The position determines how object points are projected onto the viewing sphere and the orientation $Q^{-1}$ rearranges these directions. In the sequel both orientation and position of the camera will change over time. In the continuous time case we will use the notation $(Q_t, T_t), t \in \mathbb{R}$ for the orientation and position of the camera at time $t$. In the discrete time case the camera at time t will be represented by subscripts $(Q_t, T_t), t \in \mathbb{Z}$. The image position $u_t$ of a point $U$ at time $t$ is thus

$$\lambda_t u_t = P_t U = Q_t^{-1} \left[ I \mid -T_t \right] U. \tag{5}$$

Alternatively the notation $\bar{Q}_t = Q_t^{-1}$ and $\bar{T}_t = -Q_t^{-1} T_t$ will be used. The projection is then expressed as

$$\lambda_t u_t = P_t U = \left[ \bar{Q}_t \mid \bar{T}_t \right] U. \tag{6}$$

# 3 Uniqueness of solutions

The overall goal is to calculate the camera matrices $P_t$ and reconstruct the coordinates of the objects $U$ given only the image positions $u_t$. This can however only be done up to an arbitrary choice of coordinate system in the world. For the case of known orientation we can choose the origin arbitrarily and also the overall length scale. For the case of known internal calibration there is an arbitrary choice of orientation, origin and scale. For the case of uncalibrated cameras there are 15 degrees of freedom in choosing an oriented projective coordinate system. This can be seen as a multiplication of each $P_t$ from the right with the same arbitrary $4 \times 4$ matrix $B$ and at the same time multiply each coordinate vector $U$ with $B^{-1}$. This gives $16-1$ degrees of freedom since $PB$ is only defined up to positive scale.

# 4 Simplifications in the uncalibrated case

In the uncalibrated case we can simplify the problem by partially chosing the object and image coordinate system. Two such simplifications are of special interest. These are explained in detail in [6].

## 4.1 Affine reduction

If the same three points can be seen in a subsequence of images, these can be used to simplify the problem according to the following theorem.

**Theorem 4.1.** *Let $u_t^1$, $u_t^2$ $u_t^3$ be the image of three points $U^1, U^2$ and $U^3$. Choose an object coordinate system where $U^1 = (1,0,0,0)^T$, $U^2 = (0,1,0,0)^T$ and $U^3 = (0,0,1,0)^T$. Choose image coordinate system at time $t$ so that $u_t^1 = (1,0,0)^T$, $u_t^2 = (0,1,0)^T$ and $u_t^3 = (0,0,1)^T$. Then the camera projection matrix can be written*

$$P_t = Q_t^{-1}[I \mid -T_t], \tag{7}$$

*where $Q_t$ is a diagonal matrix and $T_t$ is the position of the camera at time $t$.*

By affine alignment of three corresponding points in an image subsequence, the analysis of the remaining points can be made almost as if the cameras were calibrated and with the same rotation. The unknown elements in the diagonal matrices correspond the something called *the kinetic depths* of the three image points relative to the camera motion, cf. [10, 5]. The idea is closely related to that of affine shape [10] and relative affine structure [9]. Using projective geometry one can think of this as defining the plane through the three points as the plane at infinity and also partially locking this plane at three points, leaving two degrees of freedom.

## 4.2 Projective reduction

Further simplifications can be obtained if four or more coplanar points, e.g. a planar curve, is detected in a subsequence of images.

**Theorem 4.2.** *Let $u_t^1$, $u_t^2$, $u_t^3$ and $u_t^4$ be the image of four coplanar points $U^1$, $U^2$, $U^3$ and $U^4$ so that no three of them are collinear. Choose an object coordinate system where $U^1 = (u^1(0),0)^T$, $U^2 = (u^2(0),0)^T$, $U^3 = (u^3(0),0)^T$ and $U^4(u^4(0),0)^T$. Choose image coordinate system at time $t$ so that $u_t^1 = u^1(0)$, $u_t^2 = u^2(0)$, $u_t^3 = u^4(0)$and $u_t^4 = u^4(0)$. Then the camera projection matrix can be written*

$$P_t = \left( I \mid -T_t \right), \tag{8}$$

*where $T_t$ is the position of the camera at time $t$.*

By projective alignment of the images of at least four coplanar points the problem can thus be analysed as if both internal calibration and orientation of the camera is known at all times.

We summarise this discussion with the following: The motion of the camera can be described by the pair $(Q_t, T_t)$ or $(\bar{Q}_t, \bar{T}_t)$, where $Q_t$ (or $\bar{Q}_t$) describes the orientation of the camera and $T_t$ (or $\bar{T}_t$) describes the position of the camera at time $t$. Depending on which assumptions and simplifications we have made, matrices $Q_t$ (or $\bar{Q}_t$) lie on different manifolds:

**Traditional uncalibrated setting:** The matrices $Q_t$ are arbitrary but nonsingular and two matrices are considered equal if one is a (positive) multiple of the other. There are eight degrees of freedom.

**Affinely reduced setting:** The matrices $Q_t$ are diagonal and nonsingular and two matrices are considered equal if one is a (positive) multiple of the other. There are two degrees of freedom.

**Projectively reduced setting:** The matrices $Q_t$ are identity matrices. There are no degrees of freedom. Alternatively we may require: The matrices $Q_t$ are multiples of the identity matrix and two matrices are considered equal if one is a (positive) multiple of the other.

**Calibrated setting:** The matrices $Q_t$ are orthogonal. There are three degrees of freedom. Alternatively we may require: The matrices $Q_t$ are multiples of orthogonal matrices and two matrices are considered equal if one is a (positive) multiple of the other.

These manifolds are non-linear and have so called *Lie group* structure under matrix multiplication. The corresponding *Lie algebra* will be of importance. Unlike the Lie group the Lie algebra a *linear subspace* of three by three matrices $q$:

**Traditional uncalibrated setting:** The matrices $q_t$ are arbitrary and two matrices are considered equal if their difference is a multiple of the identity matrix. There are eight degrees of freedom.

**Affinely reduced setting:** The matrices $q_t$ are diagonal and two matrices are considered equal if their difference is a multiple of the identity matrix. There are two degrees of freedom.

**Projectively reduced setting:** The matrices $q_t$ are zero matrices. There are no degrees of freedom.

**Calibrated setting:** The matrix $q_t$ are anti symmetric. There are three degrees of freedom.

These Lie algebras are obtained from the Lie groups using the exponential map

$$Q_t = \exp(q_t). \tag{9}$$

Since two matrices in the Lie Group are considered to be equal if one is a multiple of the other, it is often convenient to choose a specific representative. One such choice of representative is to always scale the matrix so that the determinant is one. Similarly two matrices in the Lie Algebra are considered to be equal if the difference is a multiple of the identity. A unique representative can be chosen by demanding that the trace of the matrix is zero. This fits in nicely with the exponential mapping since

$$1 = \det(\exp(q)) = \exp(\mathrm{tr}(q)) = \exp(0) = 1.$$

# 5 Choice of coordinate system

In the previous section we used the option to change the coordinate system of each image to simplify the problem. Sometimes it is useful to choose a canonical object coordinate frame, to obtain a canonical coordinate representation of the reconstructed object and projection matrices.

In the calibrated case a specific coordinate system can be determined by setting orientation by $Q_0 = I$, origin by $T_0 = 0$ and overall scale by $|T_1 - T_0| = 1$.

In the uncalibrated case there are four things to consider when choosing a projective coordinate system in the reconstruction. These are (i) the position of the plane at infinity (3 d o f), (ii) the individual points at the plane at infinity (8 d o f), (iii) the origin (3 d o f) and (iv) the scale (1 d o f). One way of doing this is to lock individual points at the plane at infinity by $Q_0 = I$, lock the origin by $T_0 = 0$ and the scale by $|T_1 - T_0| = 1$. The position of the plane at infinity can be chosen by choosing a specific $Q_1$. This cannot, however, be done arbitrarily, the matrix $Q_1$ must fulfill the bilinear constraint. The question of choosing a canonical coordinate system (and thereby choosing specific plane at infinity) is simpler in the affinely and projectively reduced settings. The plane at infinity is determined by the three or more points that are used in the reduction. A canonical coordinate system can then be chosen by $Q_0 = I$, $T_0 = 0$ and$|T_1 - T_0| = 1$, similar to the calibrated case.

# 6 Multilinear forms in the discrete time case

In order to understand the multilinear constraint in the infinitesimal case, it is necessary to take a look at the corresponding constraints in the discrete time case. For a thorough treatment, see [2, 6, 12]. We start with the definition.

**Definition 6.1.** The **n:th order discrete multilinear constraints** are

$$\operatorname{rank}\begin{bmatrix} \bar{Q}_0 & \bar{T}_0 & u_0 & 0 & 0 & \dots & 0 \\ \bar{Q}_1 & \bar{T}_1 & 0 & u_1 & 0 & \dots & 0 \\ \bar{Q}_2 & \bar{T}_2 & 0 & 0 & u_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{Q}_n & \bar{T}_n & 0 & 0 & 0 & \dots & u_n \end{bmatrix} \leq n+4. \tag{10}$$

**Theorem 6.1.** *In a sequence of discrete images corresponding points with coordinates $u_0, u_1, \dots, u_n$ must obey the n:th order discrete multilinear constraints. This means that there exist a solution to* (10) *for $\bar{Q}_0, \dots, \bar{Q}_n$ and $\bar{T}_0, \dots, \bar{T}_n$ that holds for every corresponding point sequence.*

This can be seen by noticing that each projection constraint

$$\lambda_i u_i = \bar{Q}_i \mathbf{U} + \bar{\mathbf{T}}_\mathbf{i}$$

is a linear in $\mathbf{U}$ and $\lambda_0, \dots, \lambda_n$, and lining up the equations.

The above formulation is simplified by partially choosing coordinate system, such that $\bar{Q}_0 = I$ and $\bar{T}_0 = 0$ and then then eliminating $\mathbf{U}$. This is discussed in [6].

# 7 Multilinear forms in the continuous time case

The multilinear constraints in the continuous time case can be derived using Taylor series expansion of the time dependent functions in (6).

$$\lambda_t u_t = P_t U = [\bar{Q}_t \mid \bar{T}_t] U, \tag{11}$$

Using the Taylor series expansions

$$
\begin{aligned}
\lambda_t &= \lambda^0 + \lambda^1 t + \lambda^2 t^2 + \cdots, \\
\bar{Q}_t &= \bar{Q}^0 + \bar{Q}^1 t + \bar{Q}^2 t^2 + \cdots, \\
\bar{T}_t &= \bar{T}^0 + \bar{T}^1 t + \bar{T}^2 t^2 + \cdots, \\
u_t &= u_0 + u^1 t + u^2 t^2 + \cdots,
\end{aligned}
\tag{12}
$$

where $\bar{Q}^i$ are arbitrary $3 \times 3$ matrices, gives

$$
\begin{aligned}
&(\lambda^0 + \lambda^1 t + \lambda^2 t^2 + \cdots)(u^0 + u^1 t + u^2 t^2 + \cdots) = \\
&= [\bar{Q}^0 + \bar{Q}^1 t + \bar{Q}^2 t^2 + \cdots \mid \bar{T}^0 + \bar{T}^1 t + \bar{T}^2 t^2 + \cdots] U.
\end{aligned}
$$

Identifying the coefficients of $t^i$ for $i = 0, \ldots, n$ gives

$$
\begin{bmatrix}
\bar{Q}^0 & \bar{T}^0 & u^0 & 0 & 0 & \ldots & 0 \\
\bar{Q}^1 & \bar{T}^1 & u^1 & u^0 & 0 & \ldots & 0 \\
\bar{Q}^2 & \bar{T}^2 & u^2 & u^1 & u_0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\bar{Q}^n & \bar{T}^n & u^n & u^{n-1} & u^{n-2} & \ldots & u^0
\end{bmatrix}
\begin{bmatrix}
-\mathbf{U} \\ -1 \\ \lambda^0 \\ \lambda^1 \\ \lambda^2 \\ \cdots \\ \lambda^n
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ \vdots \\ 0
\end{bmatrix}.
\tag{13}
$$

Since this set of equations has a non-trivial solution the matrix cannot have full rank.

**Definition 7.1.** The **n:th order continuous constraint for the traditional setting** are

$$
\operatorname{rank}
\begin{bmatrix}
\bar{Q}^0 & \bar{T}^0 & u^0 & 0 & 0 & \ldots & 0 \\
\bar{Q}^1 & \bar{T}^1 & u^1 & u^0 & 0 & \ldots & 0 \\
\bar{Q}^2 & \bar{T}^2 & u^2 & u^1 & u^0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\bar{Q}^n & \bar{T}^n & u^n & u^{n-1} & u^{n-2} & \ldots & u^0
\end{bmatrix}
\leq n+4.
\tag{14}
$$

**Theorem 7.1.** *In a continuous sequence of images the image coordinates $u^0$ and their derivatives, $u^i$, up to order $n$ at the same instant of time must obey the n:th order continuous constraint for the traditional setting. This means that there exist a solution to (10) for $\bar{Q}^0, \bar{Q}^1, \ldots, \bar{Q}^n$ and $\bar{T}^0, \bar{T}^1, \ldots, \bar{T}^n$.*

Notice the similarity with the discrete time case. Observe that $\lambda^k = k! \frac{d^k \lambda}{dt^k}$, and similarly for the other variables. This means in particular that $u^1$ has the meaning of image velocity or optical flow.

One way to simplify the infinitesimal multilinear constraints is to choose $P^0 = [I \mid 0]$ (which implies $Q^0 = I$, $\bar{T}^0 = \mathbf{0}$) and eliminate $\mathbf{U}$. The constraint (14) is then simplified as

$$
\operatorname{rank}
\begin{bmatrix}
u^1 - \bar{Q}^1 u^0 & \bar{T}^1 & u^0 & 0 & \ldots & 0 \\
u^2 - \bar{Q}^2 u^0 & \bar{T}^2 & u^1 & u^0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
u^n - \bar{Q}_n u^0 & \bar{T}^n & u^{n-1} & u^{n-2} & \ldots & u^0
\end{bmatrix}
\leq n+2.
$$

If we want to achieve uniqueness we may require $|\bar{T}^1| = 1$ and $\operatorname{tr} \bar{Q}^i = 0$. This can be seen in the same way as in the affinely reduced case, because since $\bar{Q}_t$ is undetermined up to a scale factor, $\operatorname{tr} \bar{Q}^i$ becomes undetermined. One further complication is that $P_1$ have three degrees of freedom when $P_0 = [I \mid 0]$ have been chosen. This means that we must impose three further constraints on $\bar{Q}^1$ in order to achieve uniqueness and these constraints will fix the plane at infinity, see also [4].

# 8 Remarks concerning the infinitesimal motion constraints

Study again the simplification of the infinitesimal multilinear One use of this constraint is to calculate $(\bar{Q}^i, \bar{T}^i)$ given the the motion of the points in the image $(u^i)$. Typically the relative noise increases with increasing orders of differentiation. We therefore expect $u^2$ to be noisier than $u^1$ which in turn is noisier than $u^0$. It is therefore natural to estimate motion parameters in steps. First estimate $(\bar{Q}^1, \bar{T}^1)$ using the infinitesimal bilinear constraint

$$
\operatorname{rank} \begin{bmatrix} u^1 - \bar{Q}^1 u^0 & \bar{T}^1 & u^0 \end{bmatrix} \leq 3.
\tag{15}
$$

Using the calibrated camera model $\bar{Q}^1$ is the derivative of a rotation operator, i.e a skew-symmetric matrix, usually denoted $\Omega$. The constraint (15) is then simply the optical flow equation, and $\bar{T}^1$ is the focus of expansion. Using the uncalibrated camera model $\bar{Q}^1$ is a general matrix with trace zero. This is the constraint derived and used in [13]. The estimate of $(\bar{Q}^1, \bar{T}^1)$ can then be used to estimate $(\bar{Q}^2, \bar{T}^2)$ using the infinitesimal trilinear constraint

$$
\operatorname{rank}
\begin{bmatrix}
u^1 - \bar{Q}^1 u^0 & \bar{T}^1 & u^0 & 0 \\
u^2 - \bar{Q}^2 u^0 & \bar{T}^2 & u^1 & u^0
\end{bmatrix}
\leq 4.
\tag{16}
$$

# 9 Motion observability

## Motion observability from the bilinear constraint

Does the infinitesimal bilinear constraint determine camera motion uniquely up to choice of camera system? Using the infinitesimal bilinear constraint we can calculate $T_t'$ up to an unknown scale factor and $Q_t'$ up to an arbitrary choice of plane at infinity. Since only the direction of $T_t'$ is known at each time instant, it is not possible to reconstruct $T$ uniquely. Thus, the infinitesimal bilinear constraint is not enough to determine motion.

### Motion observability from the trilinear constraint

If $T_t'$ and $Q_t'$ are known and $T_t' \neq 0$, then $T_t''$ and $Q_t''$ are uniquely determined by the trilinear constraint.

One can think of this as $T_t''$ and $Q_t''$ being a function of $T_t'$, $Q_t'$, $T_t$, $Q_t$ and image motion at this time instant.

$$[T_t'', Q_t''] = g(T_t', Q_t', T_t, Q_t, t, u_t, u_t', u_t'')$$

It is a well known fact that these kind of differential equations can be solved at least locally, given a set of initial conditions on $T_0$, $Q_0$, $T_0'$, $Q_0'$. These initial conditions are determined by choosing coordinate system and at the same time fulfilling the bilinear constraint at $t = 0$. Thus the full motion of the camera is observable from the infinitesimal trilinear constraint if $T_t' \neq 0$.

## 10 Estimation of motion parameters using the infinitesimal multilinear constraints

There are some difficulties of using the infinitesimal multilinear constraints to estimate motion parameters. First of all it can be quite difficult to obtain good estimates of the image point positions and their derivatives. Secondly, since these estimates are noisy, noise must be modelled and taken into account. This affects the way motion parameters should be estimated.

### Using the bilinear constraint

The bilinear constraint

$$\operatorname{rank}\begin{bmatrix} u^1 - \bar{Q}^1 u^0 & \bar{T}^1 & u^0 \end{bmatrix} \leq 3, \tag{17}$$

involves the the first order derivative in camera motion, $T'$ and $q = Q^1$. One major difference between the discrete case and the infinitesimal is that the derivative of the orientation, $q$, lie on a *linear* manifold. The velocity $T'$ also lie on a linear manifold. It can, however, only be determined up to scale. If $T'$ is known, the problem of determining $q$ is linear so it can quite easily be solved with linear methods. On the other hand, if $q$ is known the problem of determining $T'$ can be formulated and solved in a linear fashion. This suggests a fast two-step iterative method. Guess $q$. Holding $q$ fixed solve for $T'$. Holding $T'$ fixed solve for $q$. This method has been tried and in most cases it does seem to converge nicely.

An alternative method could be to tessellate the sphere of directions $T' \in \mathbb{S}^2$. For each $T'$, solve for $q$ and store the residual. Choose the pair $(T', q)$ that gave the lowest residual.

Any of these two methods can be refined by taking the motion parameters as an initial estimate of $(T', q)$, in a non-linear least squares minimisation. An advantage of this refinement is that it allows for more sophisticated error measures, e.g. the maximum likelihood estimate, that takes into account the quality of estimates of $u$ and $u'$. Another advantage of this refinement is that it allows for an analysis of the stability of the solution through the analysis of the residuals at the optimum.

### Using the trilinear constraint

The trilinear constraint

$$\operatorname{rank}\begin{bmatrix} u^1 - \bar{Q}^1 u^0 & \bar{T}^1 & u^0 & 0 \\ u^2 - \bar{Q}^2 u^0 & \bar{T}^2 & u^1 & u^0 \end{bmatrix} \leq 4, \tag{18}$$

involves the first and second order derivatives in camera motion. As in the bilinear constraint, it should be possible to use a two step iterative method. Guess $(\bar{Q}^1, \bar{Q}^2)$. Holding $(\bar{Q}^1, \bar{Q}^2)$ fixed solve for $(\bar{T}^1, \bar{T}^2)$. Holding $(\bar{T}^1, \bar{T}^2)$ fixed solve for $(\bar{Q}^1, \bar{Q}^2)$. Close to a solution the method could be refined by non-linear maximisation of a Likelihood function.

## 11 Experiments

To illustrate the continuous constraints, we have used iterative methods as described above. We only consider the first order continuous constraint. An image sequence of an indoor scene have been used, see Figure 1, where one image in the sequence is shown. The whole sequence contains more than 200 images. To illustrate the applicability of the continuous constraints, we have only used 2 images. Points have been extracted using a subpixel corner detector, made available by [7], and we have used 28 points with correspondences. The correspondences are made by hand, but it is easily done automatically since the time instant between the different exposures are very small. The affinely reduced coordinates have been calculated from the images, giving $u_0$ and $u_h$, where $h$ denotes the time increment between the different exposures. In this case $h = 1/25$sec. We have used $u^0 = u_0$. The derivatives have been computed from image 1 and image 2 using a difference approximation

$$u^1 = \frac{u_h - u_0}{h}.$$

Using the iterative approach and the affinely reduced setting only 10 iterations, starting from $Q^1 = 0$, we obtain the following solution fulfilling the first order constraint:

$$\bar{q} = \bar{Q}^1 = \begin{pmatrix} 0.0490 & 0 & 0 \\ 0 & -0.0543 & 0 \\ 0 & 0 & 0.0053 \end{pmatrix}$$
$$\bar{T}^1 = (0.4838, 0.3088, -0.8189).$$

This solution can be compared to the solution obtained in the discrete case between $u_0$ and $u_h$:

$$\bar{Q}_h = \begin{pmatrix} 1.0021 & 0 & 0 \\ 0 & 0.9976 & 0 \\ 0 & 0 & 1.0003 \end{pmatrix}$$
$$\bar{T}_h = (0.5173, 0.2752, -0.8103).$$

Here $\bar{Q}_h$ should be compared to

$$I + h\bar{Q}^1 = \operatorname{diag}(1.0020, 0.9978, 1.0002)$$

and $\bar{T}_h$ to $\bar{T}^1$. The angle between $\bar{T}^1$ to $\bar{T}_h$ is 2.8 degrees.

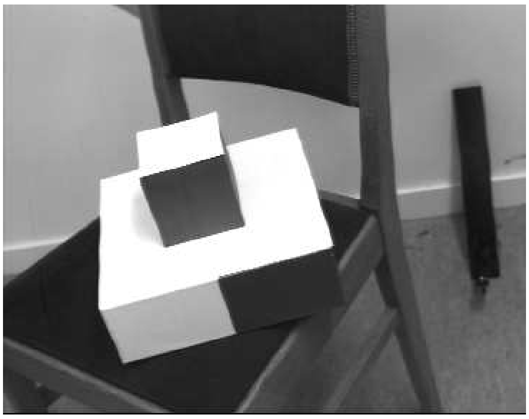

**Figure 1: One image in the sequence used in the continuous case.**

## 12 Discussion and Conclusions

The continuous constraints can be used to design filters to estimate structure and motion from image sequences. Using only the first order constraint gives just the direction of the movement of the camera. Having only this information it is not possible to build up the whole camera movement. Using the second order constraint gives the second order derivative of the camera movement with a scale consistent with the first order derivative. This information can be used to build up the camera movement. This is analogous to the discrete case, where the trilinearities must be used in order to estimate the camera movement, if only multilinear constraints between consecutive images are used.

The example above shows that the first order continuous constraint is comparable to the discrete case. However, the continuous constraints are sensitive to noise because estimating the image velocities amplifies the noise present in the images. The higher order continuous constraints are even more sensitive, because they involve higher order derivatives. Using filtration techniques to estimate the derivatives from image coordinates in more than two images could reduce the influence of noise.

Despite these drawbacks we believe that the continuous constraints are necessary to understand and deal with image sequences where the time interval between the different exposures are small.

In this paper the simplied formulation of the multilinear forms that exist between a sequence of images has been used to derive similar constraints in the infinitesimal or velocity case. The new formulation makes it easier to analyse the matching constraints in image sequences. It becomes apparent that multilinear forms contain information in the bilinear and trilinear forms only. This representation is fairly close to the representation of the motion and it is easy to generalise to different settings. Four such settings calibrated, uncalibrated, affinely reduced and projectively reduced, are described in the paper.

Much is simplified through the choice of affine basis in each image. Further simplifications can be achieved if four or more coplanar points can be identified in a short subsequence. These coplanar points can be used in a projectively reduced setting. After the reduction the analysis of the remaining points can be made as if the camera underwent a purely translating motion, yielding even simpler forms on the fundamental matrices and the trilinearities.

The continuous counterpart to the multilinear forms are derived in all four settings, showing again the similarities between the affinely reduced, the projectively reduced, the traditional uncalibrated and the calibrated settings.

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