Singular integrals, rectifiability, and the David-Semmes problem

Xavier Tolsa

Lund, 13 June 2013

Hausdorff measures in $\mathbb{R}^d$

- $\mathcal{H}^1$ = length.
- $\mathcal{H}^2$ = area.
- $\mathcal{H}^n$ = n-dimensional volume.

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For $E \subset \mathbb{R}^d$, $s \geq 0$, $\varepsilon > 0$:

$$\mathcal{H}^s_\varepsilon(E) = \inf\Big\{\sum_i \mathrm{diam}(A_i)^s : \, E \subset \bigcup_i A_i, \; \mathrm{diam}(A_i) \leq \varepsilon\Big\}.$$

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If $0 < \mathcal{H}^s(E) < \infty$, then

$$\dim_{\mathcal{H}}(E) = s.$$

Examples

- The ternary Cantor set E:

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- Sierpinski triangle:

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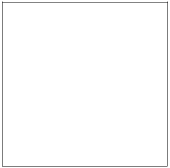

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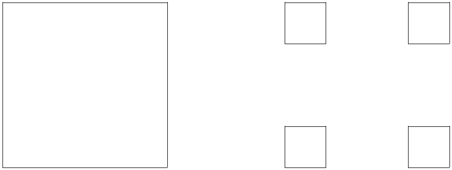

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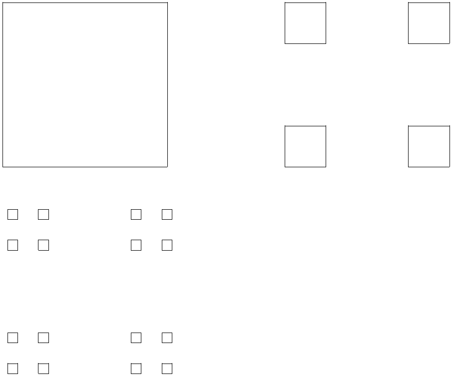

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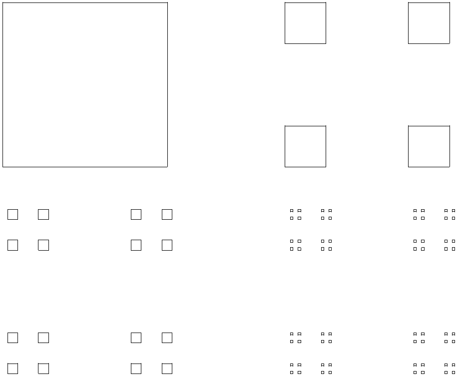

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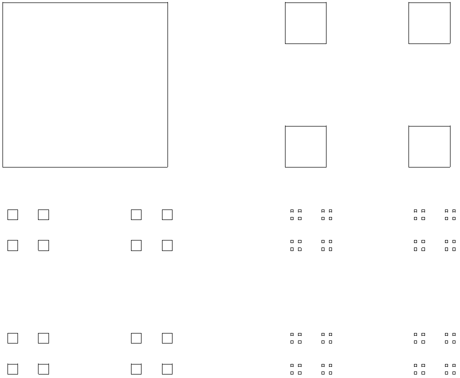

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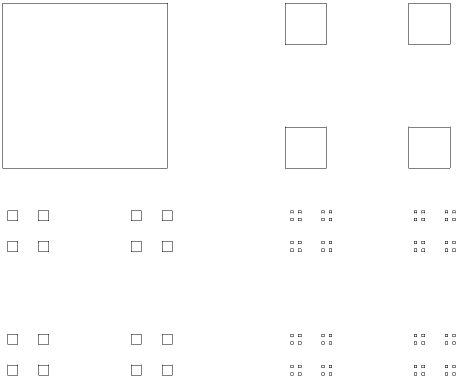

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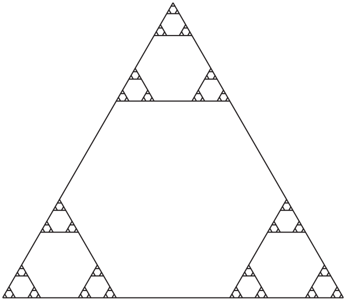

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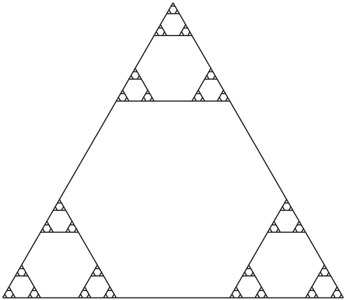

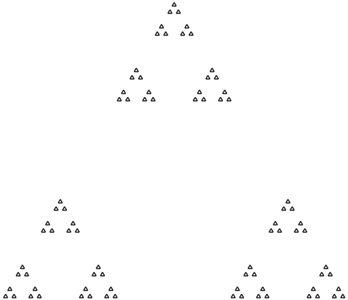

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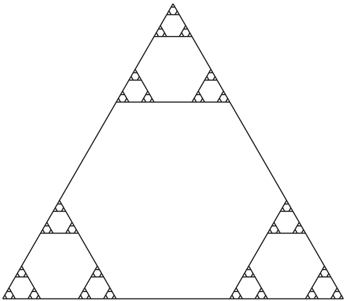

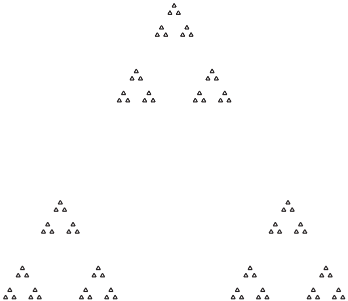

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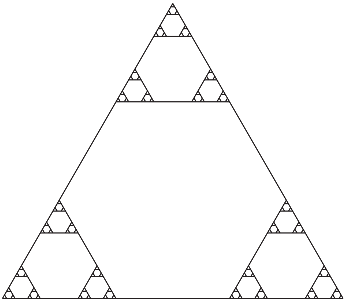

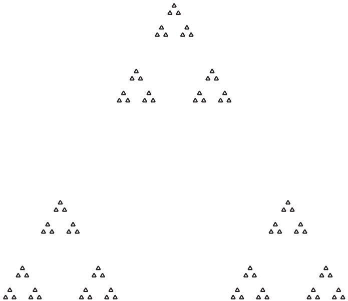

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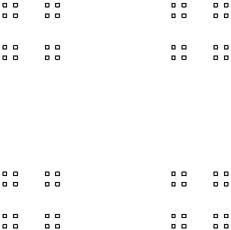

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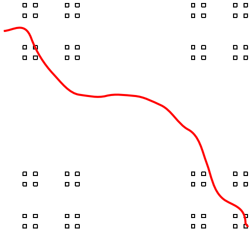

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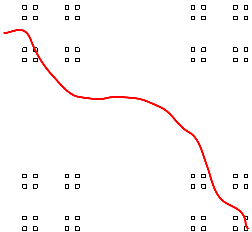
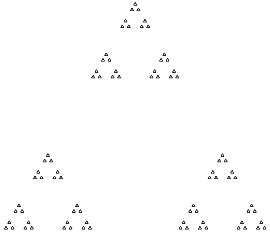
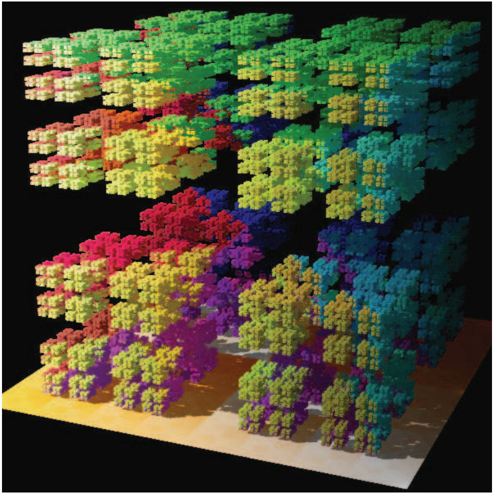




A 3-dimensional purely unrectifiable set



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- *Lower s-dimensional density:*

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Open problem (Besicovitch):

E purely unrectifiable $\Rightarrow$ $D^1_*(x,E) \leq 1/2$ $\mathcal{H}^1$-a.e. $x \in E$.

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We consider singular integrals associated with odd n-dimensional Calderón-Zygmund kernels $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$:

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The Cauchy and Riesz transforms

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- Other 1-dimensional kernels in $\mathbb{C}$: $\dfrac{z^3}{|z|^4}$, $\dfrac{z^5}{|z|^6}$, ...

Singular integrals

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(Essentially due to Mattila-Melnikov).

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Remark: The last statement is equivalent to saying that, for $\mu = \mathcal{H}^n_{|E}$,

$$\|T_K(f\mu)\|_{L^2(\mu)} \leq c\|f\|_{L^2(\mu)} \quad \text{for all } f \in L^2(\mu) \implies E \text{ is } n\text{-rectifiable.}$$

For K being the Riesz kernel this is the David-Semmes problem.

Motivation: removable singularities

Let $E \subset \mathbb{C}$ be compact. E is removable for bounded analytic functions if for every open set $\Omega \supset E$, every function $f : \Omega \setminus E :\to \mathbb{C}$ which is analytic and bounded can extended analytically to the whole Ω.

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Theorem (David, 1999)

Let $E \subset \mathbb{C}$ *with* $\mathcal{H}^1(E) < \infty$. *Then* E *is non removable for bounded analytic functions iff there exists* $F \subset E$ *with* $\mathcal{H}^1(F) > 0$ *such that, for* $\mu = \mathcal{H}^1_F$, $\mathcal{C}_*\mu(z) = \sup_{\varepsilon>0} |\mathcal{C}_\varepsilon\mu(z)| < \infty$ *for* μ*-a.e.* z.

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Removable singularities and the Cauchy and Riesz transforms

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In the case $\mathcal{H}^1(E) = \infty$ (or $\mathcal{H}^n(E) = \infty$), other results by T. (or Volberg).

Cauchy transform and rectifiability

For the Cauchy transform $\mathcal{C}\mu(z) = \int \frac{1}{z-\xi}\, d\mu(\xi)$:

Theorem (David-Léger, 1999)

Let $E \subset \mathbb{C}$, with $\mathcal{H}^1(E) < \infty$. The following are equivalent:

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- The proof uses essentially the curvature formula of Melnikov:

$$\frac{1}{R(z_1, z_2, z_3)^2} = \sum_{s \in S_3} \frac{1}{(z_{s_2} - z_{s_1})\overline{(z_{s_3} - z_{s_1})}}, \quad z_1, z_2, z_3 \in \mathbb{C},$$

where S_3 is the group of permutations of three elements.

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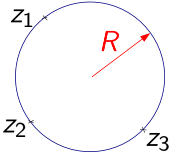

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- The result is open for $1 < n < d-1$.

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- Variational argument which uses the harmonicity of $\dfrac{x}{|x|^{n+1}}$.
- A deep geometric characterization of uniform rectifiability due to David and Semmes.

Other kernels

Other 1-dimensional kernels K in $\mathbb{C}$ such that

$$T_{K,*}\mathcal{H}^1_{|E}(x) < \infty \;\; \mathcal{H}^1\text{-a.e. } x \in E \quad \Longrightarrow \quad E \text{ is rectifiable:}$$

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- No more kernels known!!

The kernels $\dfrac{xy^2}{|z|^4}$ and $\dfrac{z^3}{|z|^4}$

[Huovinen] For $K_1(z) = \dfrac{xy^2}{|z|^4}$, there exists $E \subset \mathbb{C}$ with $0 < \mathcal{H}^1(E) < \infty$ which is purely unrectifiable, such that

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- [Jaye-Nazarov, 2013] However,

$$T_{K_2,*} \mathcal{H}^1_{|E}(x) < \infty \ \ \mathcal{H}^1\text{-a.e. } x \in E \text{ does not imply } E \text{ rectifiable!}$$

Riesz transforms of fractional homogeneity

Recall that for $0 < \mathcal{H}^s(E) < \infty$, $s \notin \mathbb{Z}$,

$$D^s(x, E) \text{ does not exist } \mathcal{H}^s - \text{a.e. } x \in E.$$

Riesz transforms of fractional homogeneity

Theorem

Let $E \subset \mathbb{R}^d$, with $0 < \mathcal{H}^s(E) < \infty$, for some $s \in (0,1) \cup (d-1,d)$. Then

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- Also, for arbitrary $s \in (0,d) \setminus \mathbb{Z}$,

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  [Ruiz de Villa-T., 2010].

Riesz transforms of fractional homogeneity

Conjecture

For $s \in (0, d) \setminus \mathbb{Z}$, given a finite measure μ in $\mathbb{R}^d$ such that $\mu(B(x, r)) \leq c\, r^s$ for all x, r,

$$\|R^s \mu\|_{L^2(\mu)}^2 \approx \sum_{Q \in \mathcal{D}} \left(\frac{\mu(3Q)}{\ell(Q)^s} \right)^2 \mu(3Q).$$

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- True for $s \in (0,1)$ [Mateu-Prat-Verdera].
- True for some Cantor sets [Mateu-T.] for all $s \in (0,d) \setminus \mathbb{Z}$.

The end

Thank you.