Old and new questions in noncommutative ring theory

Agata Smoktunowicz

University of Edinburgh
Edinburgh, Scotland, UK
e-mail: A.Smoktunowicz@ed.ac.uk

26 Nordic and 1st European-Nordic Congress of Mathematics, Lund, 10-14 June 2014

Outline

Short story of Golod-Shafarevich algebras

In 1964, Golod and Shafarevich found that, provided that the number of relations of each degree satisfies some bounds, there exist infinitely dimensional algebras satisfying given relations.

Such algebras have now come to be referred to as Golod-Shafarevich algebras.

Golod-Shafarevich algebras are used to construct infinitely dimensional algebras satisfying prescribed relations (for example nil algebras).

Definition of Golod-Shafarevich algebras

Definition

Let K be a field and let A be a free algebra in d generators. Let I be an ideal generated by relations $f_1, f_2, \ldots$ with r_i relations of degree i among $f_1, f_2, \ldots$.

Let $R = A/I$. If the series

$$(1 - dt + \sum_{i=2}^{\infty} r_i t^i)^{-1}$$

has all coefficients larger than zero then $R = A/I$ is a Golod-Shafarevich algebra.

Golod-Shafarevich theorem

Theorem

Let $H_R(t) = \sum_{i=0}^{\infty} \dim R_i t^i$ *be the Hilbert series of algebra* $R = A/I$, *with* A, I *as above. Golod and Shafarevich proved that*

$$H_R(t)(1 - dt + \sum_{i=2}^{\infty} r_i t^i) \geq 1$$

holds coefficient's-wise.

Therefore if all coefficients of $(1 - dt + \sum_{i=2}^{\infty} r_i t^i)^{-1}$ *are larger than zero then the algebra* $R = A/I$ *is infinite dimensional.*

How to check if given algebra is Golod-Shafarevich algebra?

Let K be a field and let A be a free algebra in d generators. Let I be an ideal generated by relations $f_1, f_2, \ldots$ with r_i relations of degree i among $f_1, f_2, \ldots$.

Zelmanov's survey, 1996

If there is $t_0 > 0$ such that $\sum_{i=2}^{\infty} r_i t^i$ converges at t_0 and $1 - dt_0 + \sum_{i=2}^{\infty} r_i t_0^i < 0$ then A/I is infinitely dimensional.

Remark (A.S)

If for infinitely many m, there is $0 < t_m$ such that $1 - dt_m + \sum_{i=2}^{m} r_i t_m^i < 0$ then algebra A/I is finitely dimensional.

Proof of remark (1)

Fix number m and let I_m be the ideal of A generated by all elements from f_i which have degrees not exceeding m.

If there is $t_m > 0$ such that

$$1 - dt_m + \sum_{i=2}^{m} r_i {t_m}^i < 0$$

then by the above remark of Zelmanov A/I_m is infinitely dimensional.

Proof of remark (2)

Since I is a graded ideal and

$$I = \bigcup_{i=2}^{\infty} I_m$$

it follows that A/I is infinite dimensional.

Indeed, a graded and finite dimensional algebra is nilpotent. If A/I is finitely dimensional them $A(n) \in I$ for some n, and since I is graded $A(n) \subseteq I_n$, so A/I_n is finitely dimensional, a contradiction.

Applications of Golod-Shafarevich algebras

- General Burnside Problem (Group Theory)
- Kurosh Problem (Ring Theory)
- Tower of Fields Problem (Number Theory)
- Recently: Topology

Active research area

Golod-Shafarevich groups.

Vaden, a student of Zelmanov, if there is less than $\frac{d^2}{25}$ quadratic relations then infinitely many subalgebras R^n of $R = A/I$ are Golod-Shafarevich algebras.

Open Question

Is the inverse of Golod-Shafarevich theorem true? Not generated in degree one-not true (Anick). In general open. Interesting results Wisilceny, recently Iyudu and Shkarin.

Can we assume something more about these algebras?

Golod-Shafarevich algebras are used to construct infinitely dimensional algebras satisfying prescribed relations.

Are there **domains** satisfying given relations?

Are there **PI** algebras satisfying given relations?

Are there **Noetherian algebras** satisfying given relations?

Golod-Shafarevich algebras and nil algebras

There are Golod-Shafarevich algebras which are nil.

- Nil algebras which are commutative are nilpotent.
- Nil algebras which are Noetherian are nilpotent
- Nil algebras which satisfy polynomial identity are nilpotent.

In general it is not possible to assure that there is a domain, or a polynomial identity algebra or Noetherian algebra satisfying given number of relations of each degree, even small (infinite dimensional algebra).

New result

Theorem (A.S., J.Algebra, 2013)

Given set of homogeneous relations and a finite number of relations and less than $2^{n/8}$ relations of degree n, and relations are of scarce degrees. Then we can construct prime, graded algebras with linear growth satisfying the prescribed relations.

Remark

By Warfield-Small theorem, prime, finitely generated algebras with linear growth are Noetherian algebras which satisfy polynomial identity.

Remark

Polynomial Identity algebras are 'close' to commutative algebras.

Precise formulation

- Let K be an algebraically closed field, and let A be the free noncommutative algebra generated in degree one by elements x, y.
- Let ξ be a natural number. Let I denote the ideal generated in A by homogeneous elements $f_1, f_2, \ldots, f_\xi \in A$.
- Suppose that there are exactly r_i elements among $f_1, f_2, \ldots, f_\xi$ with degrees larger than 2^i and not exceeding 2^{i+1}.
- Assume that there are no elements among $f_1, f_2, \ldots, f_\xi$ with degree k if $2^n - 2^{n-3} < k < 2^n + 2^{n-2}$ for some n.
- Denote $Y = \{n : r_n \neq 0\}$. Suppose that for all $n \in Y$, $m \in \{0\} \cup Y$ with $m < n$ we have

$$2^{3n+4} r_m^{32} < r_n < 2^{2^{n-m-3}}.$$

Assertion

Then A/I contains a free noncommutative graded subalgebra in two generators.

These generators are monomials of the same degree. In particular, A/I is not Jacobson radical.

Moreover, A/I can be homomorphically mapped onto a graded, prime, Noetherian algebra with linear growth which satisfies a polynomial identity.

General construction

Let K be a field and A be a free K-algebra generated in degree one by two elements x, y.

Suppose that subspaces $U(2^m)$, $V(2^m)$ of $A(2^m)$ satisfy, for every $m \geq 1$, the following properties:

1. $V(2^m)$ is spanned by monomials;
2. $V(2^m) + U(2^m) = A(2^m)$ and $V(2^m) \cap U(2^m) = 0$;
3. $A(2^{m-1})U(2^{m-1}) + U(2^{m-1})A(2^{m-1}) \subseteq U(2^m)$;
4. $V(2^m) \subseteq V(2^{m-1})V(2^{m-1})$,

   where for $m = 0$ we set $V(2^0) = Kx + Ky$, $U(2^0) = 0$.

Ideal E

We define a graded subspace E of A by constructing its homogeneous components $E(k)$ as follows.
Given $k \in N$, let $n \in N$ be such that $2^{n-1} \leq k < 2^n$.
Then $r \in E(k)$ precisely if, for all $j \in \{0, \ldots, 2^{n+1} - k\}$, we have

$$A(j)rA(2^{n+1} - j - k) \subseteq U(2^n)A(2^n) + A(2^n)U(2^n).$$

More compactly,

$$E(k) = \{r \in A(k) \mid ArA \cap A(2^{n+1}) \subseteq U(2^n)A(2^n) + A(2^n)U(2^n)\}. \tag{1}$$

Set then $E = \bigcup_{k \in N} E(k)$.

Ideal E

Lemma

- *The set E is an ideal in A.*
- *Moreover, if all sets $V(2^n)$ are nonzero, then algebra A/E is infinite dimensional over K.*

More on construction

1. If we have given relation f_i of degree $< 2^n$, then we can assure that elements of degree 2^{n+1} in Af_iA is a subset of

$$U(2^n)A(2^n) + A(2^n)U(2^n)$$

2. This imply that relation $f_i = 0$ holds in A/E

3. If we have finite number of relations, then from some point we will have no restrictions on sets $U(2^n)$, $V(2^n)$. We can assume $V(2^n)$ are generated by one element from some point-then A/E is a *PI (polynomial identity) algebra with linear growth*.

More on construction

1. Since we have no restrictions on sets $V(2^n)$ and $U(2^n)$ from some point, we can assume that $V(2^{n+1}) = V(2^n)V(2^n)$ for all $n > N$.

2. Then elements of $V(2^N)$ will generate free subalgebra in A/E.

3. The construction of sets $U(2^n)$ and $V(2^n)$ for $n < N$ will assure that all prescribed relations hold in algebra A/E.

New result

Theorem

Let K be a field and A be a free K-algebra generated in degree one by two elements x, y. Suppose that subspaces $U(2^m), V(2^m)$ of $A(2^m)$ satisfy properties 1-4 above, and moreover that there is n such that

$$\dim V(2^n) = 2, \quad V(2^{m+1}) = V(2^m)V(2^m)$$

for all $m \geq n$.
Then, the algebra A/E contains a free noncommutative algebra in 2 generators, and these generators are monomials of the same degree.

Recent results on free subalgebras

Anick proved that finitely presented monomial algebras with exponential growth always contain free noncommutative subalgebras.

Recently Bell and Rogalski proved that quotients of affine domains with Gelfand-Kirillov dimension two over uncountable, algebraically closed fields contain free noncommutative subalgebras in two generators.

Recent results on free subalgebras

Recent results on free subalgebras

An open question by Anick asks whether all division algebras of exponential growth contain free noncommutative subalgebras in two generators.

Recent results on free subalgebras

An open question by Anick asks whether all division algebras of exponential growth contain free noncommutative subalgebras in two generators.

Recently Bell and Rogalski answered this question in the affirmative in the case of algebras over C-the field of complex numbers.

Recent results on free subalgebras

An open question by Anick asks whether all division algebras of exponential growth contain free noncommutative subalgebras in two generators.

Recently Bell and Rogalski answered this question in the affirmative in the case of algebras over C-the field of complex numbers.

Related questions concerning Golod-Shafarevich groups have also been studied. In particular, Zelmanov proved that a pro-p group satisfying the Golod-Shafarevich condition contains a free non abelian pro p-group.

Makar-Limanov Question and nil rings

Theorem (A.S., 2009, Advances in Mathematics)

There are nil algebras such that when we extend their fields of scalars, they will contain free noncommutative algebras in two generators.

In particular, there are finitely generated algebras which does not contain free subalgebras in free generators, which after extending fields of scalars will contain free subalgebras in two generators.

This answered a question of Makar-Limanov.

Completely different situation

The situation in case of domains is completely different!

Theorem (Recent results by Bell and Rogalski)

Let R be a finitely generated algebra over uncountable field, which is a domain.

If Q does not contain a free noncommutative subalgebra in two generators, then there is a subfield F of Q and an element x in Q which is left algebraic over F.

More on free subalgebras

Theorem 1 shows that if we have a finite number of relations and less than $2^{n/8}$ relations of degree n, and relations are of scarce degrees then free algebra subject to these relations contains free noncommutative subalgebras.

Therefore, there are no nil or Jacobson radical algebras which are factor of free algebras by the the ideal generated by r_i relations of degree i, if r_i satisfy assumptions of Theorem 1 (A.S., 2013, J.Algebra).

Big open questions

Question 1

Is there a finitely presented nil algebra?

Finitely presented algebra is a factor of a free algebra by a finitely generated ideal.

Remark

Zelmanov noted that Question 1 is related to the big open question in Group theory, namely to the Burnside Question for finitely presented groups.

Big problems

Previously we only assumed that our prescribed relations are homogeneous.
The case when prescribed relations are not homogeneous is related to some main open problems in noncommutative ring theory, group theory and noncommutative algebraic geometry.

Kurosh problem for domains

Is a finitely generated algebraic algebra A which is a domain always finite dimensional ?

Latyshev problem

Is a finitely generated ring, which is a division ring necessarily finite?

Some known results

Wedderburn's theorem

Finite division rings are commutative.

Jacobson's theorem

Algebraic division algebras over finite fields are commutative.

The Jacobson,Kaplansky, Levitzki, Shirshov result

Algebraic algebras of bounded degree are finite dimensional.

Amitsur's result

Finitely generated division algebras over uncountable fields are algebraic.

Noncommutative singularities

Recently many open questions have arisen about algebras satisfying a prescribed number of relations in the area of resolutions of noncommutative singularities.

The following question related to equivalences of the derived category of 3-folds in algebraic geometry was posed by Donovan and Wemyss.

The question of Donovan and Wemyss

Suppose that F is the formal free algebra in two variables, and consider two relations f_1, f_2 such that if we write both f_1 and f_2 as a sum of words, each word has degree two or higher.

Denote I to be the two sided ideal generated by f_1 and f_2.

Is it true that if F/I is finite dimensional, it cannot be commutative?

Answer (and a slightly more general result)

Lemma (A.S.)

Let K be a field.

Let F be either the free associative K- algebra on the set of free generators $X = \{x_1, x_2, \ldots, x_n\}$ over the field K or F be the formal free power series algebra over K in n variables $x_1, \ldots, x_n$.

Let $d \leq \frac{n(n-1)}{2} + 1$.

Consider d relations $f_1, f_2, \ldots, f_d \in F$ such that if we write each of the $f_1, \ldots, f_d$ as a sum of words then each word has degree two or higher.

If F/I is finite dimensional then it cannot be commutative.

Recent results

Theorem (A.S, L. Bartholdi, 2013)

It is possible to construct algebras with polynomial growth satisfying given relations if the numbers of relations of each degree is small enough.

Theorem (A.S, 2009)

Images of some Golod-Shafarevich algebras have always exponential growth.

Theorem 1 and Theorem 2 answered questions by Zelmanov.

Questions on growth of algebras

Questions on growth of algebras

Question 1

Can we construct algebras with small growth satisfying given relations, provided that the number of relations of each degree is not too big?

Questions on growth of algebras

Question 1

Can we construct algebras with small growth satisfying given relations, provided that the number of relations of each degree is not too big?

Question 2

Can we construct algebras with finite Gelfand-Kirillov dimension?

Questions on growth of algebras

Question 1

Can we construct algebras with small growth satisfying given relations, provided that the number of relations of each degree is not too big?

Question 2

Can we construct algebras with finite Gelfand-Kirillov dimension?

Question 3

Which functions are growth functions of algebras?

Questions on growth of algebras

Question 1

Can we construct algebras with small growth satisfying given relations, provided that the number of relations of each degree is not too big?

Question 2

Can we construct algebras with finite Gelfand-Kirillov dimension?

Question 3

Which functions are growth functions of algebras?

Question 4

Why should it interest us?

Gelfand-Kirillov dimension

1. Gelfand-Kirillov dimension was introduced by Gelfand and Kirillov to solve open question on Lie algebras.

2. Gelfand-Kirillov dimension has applications in noncommutative algebraic geometry.

3. Gromov theorem says that if G is a finitely generated group and the group algebra $K[G]$ has finite Gelfand-Kirillov dimension then G is nilpotent by finite. Hence torsion group of finite Gelfand-Kirillov dimension is finite.

4. Growth of groups have been extensively studied and there are some basic open questions in this area.

Positive results

Theorem (A.S, L. Bartholdi, Quarterely J.Math., 2013)

It is possible to construct algebras with polynomial growth satisfying given relations if the numbers of given relations of each degree is small enough.

Theorem (A.S, Glasgow J.Math., 2009)

Images of some Golod-Shafarevich algebras have always exponential growth.

Theorem 1 and Theorem 2 answered questions by Zelmanov.

Golod-Shafarevich groups versus algebras

Results on Golod-Shafarevich groups and algebras often mirror each other but proofs are different.

Groups

Ershov-there exists Golod-Shafarevich groups without infinite images of polynomial growth (in fact there exists Golod-Shafarevich algebras satisfying property tau).

Algebras

(A.S.): Generic Golod-Shafarevich algebras with exponential number of defining relations of scarce degrees have all infinitely dimensional modules, and homomorphic images of exponential growth.

Homomorphic images

Remark

If the number of defining relations of a GS algebra grows polynomially with respect to the degree, then there are homomorphic images of polynomial growth (A.S+L.B).

Golod-Shafarevich groups versus algebras

Theorem (Zelmanov, 2000)

Golod-Shafarevich groups contain non-abelian free pro-p-groups.

Related results on free subgroups of Golod-Shafarevich groups were obtained by Kassabov.

Theorem (A.S., 2012)

Finitely presented algebras with a limited number of defining relations contain free subalgebras.

Question 1

Which growth functions are possible for growth of algebras?

Remark

Growth function should not depend of the generating set. Therefore, by analogy with group theory it is assumed that two functions $f(x), g(x)$ are equivalent if there is number C such that $f(x) < g(Cx)$ and $g(x) < f(Cx)$.

Question 1

Theorem (A.S+Bartholdi, Quarterly J.Math, 2013)

Let f be submultiplicative and increasing, that is, $f(m+n) \leq f(m)f(n)$ for all m, n, and $f(n+1) \geq f(n)$. Then there exists a finitely generated algebra B whose growth function $v(n)$ satisfies

$$f(2^n) \leq \dim B(2^n) \leq 2^{2n+3} f(2^{n+1}).$$

Furthermore, B may be chosen to be a monomial algebra.

Remark

This implies that any sufficiently regular function that grows at least as fast as $n^{\log n}$ can be growth function of an algebra.

New interesting questions

Question 2

Which assumptions to add to generalize the Golod-Shafarevich theorem in the case of ungraded relations?

Question 3

When algebra generated by one relation is a domain?

Question 4

When algebra generated by small number of relations can be mapped onto a domain? What assumptions about generating relations should we add?

More open problems

Question 5 (Amberg, Kazaring, 1998)

Let A be a finitely generated nil algebra which is not nilpotent. Can the adjoint group of A be finitely generated?

Question 6 (Bartholdi, 2011)

What is growth of an adjoint group of an algebra?

Grigorchuk showed that there exist semigroups of growth strictly between polynomial and exponential.

One of the tantalizing open problems is the existence of groups of intermediate growth strictly between polynomial and $\exp(n^{1/2})$.

Thank you very much!

Please answer the posed questions, if possible!