Polynomials on Polydiscs

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We will be interested in polynomials F in d complex variables:

$$F(z) = \sum_{\alpha} a_\alpha z^\alpha;$$

here $z = (z_1, ..., z_d)$, $\alpha = (\alpha_1, ..., \alpha_d)$ is a multi-index with α_j nonnegative integers, and

$$z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d}.$$

We will link such polynomials to analysis on polydiscs by considering L^p norms (and more general L^p integrals) with respect to normalized Lebesgue measure σ_d on the unit polycircle $\mathbb{T}^d$. In particular, we set

$$\|F\|_p := \begin{cases} \left(\int_{\mathbb{T}^d} |F(z)|^p d\sigma_d(z)\right)^{1/p}, & 1 \le p < \infty \\ \sup_{z \in \mathbb{T}^d} |F(z)|, & p = \infty. \end{cases}$$

Elementary facts

We should bear in mind that

- $\|F\|_2^2 = \sum_\alpha |a_\alpha|^2$ (by orthonormality of z^α)
- $\|F\|_p \leq \|F\|_q$ when $p \leq q$ (by Hölder's inequality)
- $\|F\|_\infty = \sup_{z \in \mathbb{D}^d} |F(z)|$ (by maximum principle).

The polycircle $\mathbb{T}^d$ is referred to as the distinguished boundary of the polydisc $\mathbb{D}^d$.

Hardy spaces on $\mathbb{T}^d$

The three main results to be presented in this talk are all about polynomials on polydiscs. We will, however, also need to consider the closely related Hardy spaces $H^p(\mathbb{T}^d)$; for $1 \le p < \infty$, this is the closure of the set of polynomials with respect to the norm $\|\cdot\|_p$, and $H^\infty(\mathbb{T}^d)$ is the space of bounded holomorphic functions on the unit polydisc $\mathbb{D}^d$.

As a background for the three main results to be presented, I will discuss briefly three different perspectives that we may take when trying to understand polynomials and more generally spaces of holomorphic functions (such as H^p) on polydiscs.

First perspective: From $d = 1$ to $d > 1$

Rudin's approach (from the preface of his 1969 "Function Theory in Polydiscs"): "*Briefly, the object is to see how much of our extremely detailed knowledge about holomorphic functions in the unit disc [...] can be carried over to an analogous situation in several variables, namely to polydiscs*."

This suggests both purely function theoretic problems (studied in depth by Rudin) and how the extensive interplay between function theory and operator theory on the unit disc carries over to the unit polydisc.

Second perspective: From $d < \infty$ to $d = \infty$

This is Harald Bohr's viewpoint, arising in his study of Dirichlet series. Roughly, this is about the asymptotics when $d \to \infty$ or results that do not depend on the dimension d.

Bohr was motivated by the following basic observation: Let $f(s) = \sum_{n\geq 1}^{N} a_n n^{-s}$ be an ordinary Dirichlet polynomial. We factor each integer n into a product of prime numbers $n = p_1^{\alpha_1} \cdots p_d^{\alpha_d}$ and set $z = (p_1^{-s}, ..., p_d^{-s})$. Then

$$f(s) = \sum_{n=1}^{N} a_n (p_1^{-s})^{\alpha_1} \cdots (p_d^{-s})^{\alpha_d} = \sum_{n=1}^{N} a_n z_1^{\alpha_1} \cdots z_d^{\alpha_d} =: F(z).$$

Bohr's correspondence is not just formal: thanks to a classical result of Kronecker on diophantine approximation, we have e.g.

$$\sup_{t\in\mathbb{R}} |f(it)| = \sup_{z\in\mathbb{T}^{\pi(N)}} F(z).$$

Third perspective: Independent Steinhaus variables

We may interpret the interaction between the independent variables $z_1, z_2, ..., z_d$ on $\mathbb{T}^d$ in probabilistic terms, now viewing the measure σ_d as a probability measure. A random variable with a uniform distribution on the unit circle $\mathbb{T}$ is known as a Steinhaus variable; our polynomial F is in this context known as a polynomial chaos with respect to d independent Steinhaus variables.

- In the simplest (linear) case, when $F(z) = a_1 z_1 + \cdots + a_d z_d$, Khinchin's inequality says that $\|F\|_p \simeq \|F\|_2$ with constants that do not depend on d (best constants obtained by Sawa (1985) and König (2013))
- L^p norms could potentially give interesting information about the distribution of polynomial chaos
- Polynomial chaos arising from Dirichlet polynomials via Bohr's lift seems interesting but is not easy to handle.

First result: Factorization

A simple and important fact in the case $d = 1$ is that every f in $H^1(\mathbb{T})$ can be factored as

$$f = gh, \ \text{ where } \ g, h \in H^2(\mathbb{T}) \ \text{ and } \ \|g\|_2^2 = \|h\|_2^2 = \|f\|_1.$$

Reason: we may factor out the zeros of f such that $f = B\Phi$, where B is a so-called Blaschke product and Φ is without zeros. Then $\|f\|_1 = \|\Phi\|_1$, and it suffices to set $g = B\Phi^{1/2}$ and $h = \Phi^{1/2}$.

Nehari's theorem

In general, functions in several variables do not factor in this way, but operator theory helps us see what is the right analogue. Namely, the most important application of the factorization of H^1 functions is to establish Nehari's theorem (1957), which, in the case $d = 1$, characterizes bounded Hankel forms $H_\psi : H^2(\mathbb{T}^d) \times H^2(\mathbb{T}^d) \to \mathbb{C}$, defined as

$$H_\psi(f, g) = \langle fg, \psi \rangle_{L^2(\mathbb{T}^d)},$$

where $\psi \in H^2(\mathbb{T}^d)$ is the symbol of the form.

Theorem (Nehari)

H_ψ is bounded on $H^2(\mathbb{T}) \times H^2(\mathbb{T})$ if and only if ψ defines a bounded linear functional on $H^1(\mathbb{T})$.

Proof: The sufficiency is obvious; the necessity follows immediately from the factorization of functions in $H^1(\mathbb{T})$.

Weak factorization

To establish Nehari's theorem for $d > 1$, it would be enough to have $f = \sum_{j=1}^{\infty} g_j h_j$ with

$$\sum_{j=1}^{\infty} \|g_j\|_2 \|h_j\|_2 \leq C \|f\|_1$$

for a constant C independent of f. If this holds, we say that $H^1(\mathbb{T}^d)$ admits weak factorization and set

$$\|f\|_{w,1} := \inf_{\sum_j g_j h_j = f} \sum_{j=1}^{\infty} \|g_j\|_2 \|h_j\|_2.$$

Theorem (Ferguson–Lacey (2002), Lacey–Terwilleger (2009))

$H^1(\mathbb{T}^d)$ admits weak factorization for every $d > 1$.

Interpretation of Nehari's theorem

Nehari's theorem (for $d < \infty$) yields two equivalent statements:

- H_ψ is a bounded Hankel form if and only if there exists a function φ in $L^\infty(\mathbb{T}^d)$ such that the orthogonal projection of φ onto $H^2(\mathbb{T}^d)$ coincides with ψ.
- $H^1(\mathbb{T}^d)$ admits weak factorization.

From $d = 1$ to $d = \infty$: Multiplicative Hankel forms

If $\psi \in H^2(\mathbb{T})$, then the corresponding Hankel form can be written in discrete form as

$$H_\psi(a,b) = \sum_{j,k=0}^{\infty} a_j b_k \rho_{j+k}$$

where $\psi(z) = \sum_{n=0}^{\infty} \rho_n z^n$. It is bounded whenever $|H_\psi(a,b)| \leq M\|a\|_2\|b\|_2$. Nehari's theorem says that every bounded Hankel form is such that $\rho_n = \hat{\varphi}(n), n \geq 0$, where $\varphi \in L^\infty(\mathbb{T})$. The infinite dimensional version of a Hankel form is equivalent to

$$H_\psi(a,b) = \sum_{j,k=1}^{\infty} a_j b_k \rho_{j \cdot k}.$$

For each $n \in \mathbb{N}$, one considers the decomposition $n = p_1^{\alpha_1} \cdots p_d^{\alpha_d}$ in prime numbers and we associate with n as before the multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$. Then $\rho(n) = \hat{\psi}(\alpha)$.

A multiplicative version of Nehari's theorem?

Clearly, if $\psi \in L^\infty(\mathbb{T}^\infty)$ and $\rho(n) = \hat{\psi}(\alpha_1, \ldots, \alpha_d)$, then the multiplicative bilinear Hankel form is bounded. The question asked by Helson (2005) is whether the converse is true, i.e., whether for every bounded multiplicative bilinear form there is a bounded function on $\mathbb{T}^\infty$ that defines the form.

Helson proved that if the bilinear form is Hilbert-Schmidt, i.e. $\sum \rho(n)^2 d(n) < \infty$, where $d(n)$ is the number of divisors of n, then such a bounded function exists.

The Schur test

Helson, in his last paper published posthumously in 2010, used the Schur test to estimate the norm of H_ψ. In our case, all we need is a sequence of positive numbers c_j such that

$$\sum_k \rho_{jk} c_k \le B c_j.$$

Then H_ψ will be bounded on $H^2 \times H^2$ with norm $\le B$.

A key point is this: If we view H_ψ as a linear functional acting on H^1 and equip H^1 with the norm $\|\cdot\|_{w,1}$, then this functional will have the same bound B on its norm.

Helson used the following simple and natural form: $\rho_n = 1$ for $n \le N$ and $\rho_n = 0$ otherwise.

Helson's conjecture

By his simple argument involving the Schur test, Helson obtained a large class of Dirichlet polynomials Q for which $\|Q\|_2 \leq 2\|Q\|_{w,1}$, namely those with unimodular coefficients. To give a negative answer to the question of whether weak factorization extends to the infinite-dimensional polydisc, which was his main object, he "only" needed a sequence of polynomials Q_N for which $\|Q_N\|_1 = o(\|Q_N\|_2)$ when $N \to \infty$. He made the following plausible conjecture:

$$\|\sum_{n=1}^{N} n^{-s}\|_1 = o(\sqrt{N}).$$

The conjecture is a very interesting but difficult number theoretic/probabilistic problem. (I think it is best understood in probabilistic terms as a particular instance of polynomial chaos.)

Solution to Helson's problem

Although the discrete representation of multiplicative Hankel forms suggests a strong link to number theory, we can avoid the difficulties encountered by Helson, while still using the Schur test in the same way. Namely, consider instead the homogeneous polynomial

$$\psi(z) = (z_1 + z_2) \cdot (z_3 + z_4) \cdots (z_{2k-1} + z_{2k}).$$

Theorem (Ortega-Cerdà–Seip (2012))

$\sup_{f \neq 0, f \in H^1(\mathbb{D}^{2k})} \|f\|_{w,1}/\|f\|_1 \geq (\pi^2/8)^{k/2}$.

The proof shows that this particular ψ has optimal weak factorization $\psi \cdot 1$ (a function that "factors" badly!) and thus $\|\psi\|_{w,1} = \|\psi\|_2$. Our conclusion is that Nehari's theorem does not extend to $d = \infty$ or, equivalently, it does not extend to multiplicative Hankel forms.

Second result: The Bohnenblust–Hille inequality

The Bohnenblust–Hille inequality (1931) estimates the size of the coefficients (measured in terms of a suitable ℓ^p norm) of a polynomial F in terms of $\|F\|_\infty$, with a constant that *does not depend on the dimension d*.

Keep in mind that for a general polynomial $F(z) = \sum a_\alpha z^\alpha$, we trivially have

$$\Big(\sum_\alpha |a_\alpha|^2\Big)^{\frac{1}{2}} \le \|F\|_\infty.$$

The Bohnenblust–Hille inequality

We now restrict ourselves to m-homogeneous polynomials $F(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$ and ask: Is it possible to have

$$\Big(\sum_{\|\alpha\|=m} |a_\alpha|^p\Big)^{\frac{1}{p}} \le C\|F\|_\infty$$

for some $p < 2$ with C depending on m but *not* on d?

Bohnenblust–Hille

YES, and $2m/(m+1)$ is the smallest possible p.

It is of basic interest also to know the asymptotic behavior of C when $p = 2m/(m+1)$ and $m \to \infty$; Bohnenblust and Hille obtained a super-exponential bound roughly as $m^{m/2}$.

Polarization: the crux of the matter

The profound insight of the work of Bohnenblust–Hille is that there is a one-to-one correspondence (called polarization) between *m*-homogeneous polynomials and symmetric *m*-linear forms that essentially preserves L^∞ norms. (Polarization has later found many important applications.)

It is easier to do estimates with forms; in fact, the transformation from homogeneous polynomials to symmetric multilinear forms seems to be an indispensable tool for proving the Bohnenblust–Hille inequality. (The ideas for dealing with multilinear forms come from earlier work of Littlewood (1930) on corresponding estimates for bilinear forms.)

The polynomial BH-inequality is hypercontractive

The following is a recent improvement of the Bohnenblust–Hille inequality, based on a new way of connecting estimates for symmetric multilinear forms and homogeneous polynomials:

Theorem (Defant, Frerick, Ortega-Cerdà, Ounaïes, Seip 2011)

Let m and d be positive integers larger than 1. Then we have

$$\Big(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \le e\sqrt{m}(\sqrt{2})^{m-1} \sup_{z\in\mathbb{D}^d}\Big|\sum_{|\alpha|=m} a_\alpha z^\alpha\Big|$$

for every m-homogeneous polynomial $\sum_{|\alpha|=m} a_\alpha z^\alpha$ on $\mathbb{C}^d$.

Diniz–Muñoz-Fernández–Pellegrino–Seoane-Sepúlveda (2012) and others have later studied this and other similar constants. It remains unknown whether the constant grows sub-exponentially, but for the applications that I am aware of, exponential growth is good enough.

Third result: GCD sums from Poisson integrals

Recall the Poisson kernel on $\mathbb{D}$:

$$P_1(w,z) = \frac{1-|w|^2}{|1-\overline{w}z|^2}.$$

The Poisson integral

$$u(w) = \int_{\mathbb{T}} f(z) P_1(w,z)\, d\sigma_1(z)$$

gives a harmonic function whose radial limits on $\mathbb{T}$ coincide a.e. with f if say f is integrable. Choose in particular $f(z) = z^n$ and $w = r > 0$. Then

$$r^n = \int_{\mathbb{T}} f(z) P_1(r,z)\, d\sigma_1(z) = \int_{\mathbb{T}} \overline{f(z)} P_1(r,z)\, d\sigma_1(z).$$

GCD sums from Poisson integrals ctd.

An even more elaborate way of writing this, is:

$$r^{|m-n|} = \int_{\mathbb{T}} z^m \overline{z}^n P_1(r, z) d\sigma_1(z).$$

If we lift this to the polydisc, we get something interesting!

GCD sums from Poisson integral ctd.

We write

$$P_d(w,z) = \prod_{k=1}^{d} \frac{1-|w_k|^2}{|1-\overline{w_k} z_k|^2}.$$

It is convenient in this definition to allow w to be a point in the infinite-dimensional polydisc $\mathbb{D}^\infty$. Let $\beta = (\beta^{(1)}, ..., \beta^{(d)}, 0, 0, ...)$ and $\mu = (\mu^{(1)}, ..., \mu^{(d)}, 0, 0, ...)$ be multi-indices, and set $|\beta - \mu| = (|\beta^{(1)} - \mu^{(1)}|, ..., |\beta^{(d)} - \mu^{(d)}|, 0, 0, ...)$. Then if $r = (r_1, r_2, ...)$ is a sequence of positive numbers in $\mathbb{D}^\infty$, we get

$$r^{|\beta-\mu|} = \int_{\mathbb{T}^d} z^\beta \overline{z}^\mu P_d(r,z) d\sigma_d(z).$$

It leads to the following lemma.

Lemma on Poisson integrals

Lemma

For a positive sequence r in $\mathbb{D}^\infty$, arbitrary multi-indices $\beta_1 = \beta_1, \ldots, \beta_N$ with $d = \max \bigcup_j \operatorname{supp} \beta_j$ and complex numbers $c_1, \ldots, c_N$, we have

$$\sum_{k,\ell=1}^{N} r^{|\beta_k - \beta_\ell|} c_k \overline{c}_\ell = \int_{\mathbb{T}^d} \Big| \sum_{j=1}^{N} c_j z^{\beta_j} \Big|^2 P_d(r,z) d\sigma_d(z).$$

Integers in multi-index notation

If we let $p = (p_1, p_2, ...)$ denote the sequence of primes written in ascending order, then we may associate with every integer n a unique multi-index β and write $n = p^{\beta}$; setting $n_k = p^{\beta_k}$ and $n_\ell = p^{\beta_\ell}$ and taking into account that

$$|a - b| = a + b - 2\min(a, b),$$

then we get

$$(p_j^{-1})^{|\beta_k - \beta_\ell|} = \frac{(\gcd(n_k, n_\ell))^2}{n_k n_\ell}.$$

More generally, if $r = p^{-\alpha} := (p_1^{-\alpha}, p_2^{-\alpha}, ...)$ for some $\alpha > 0$, then

$$r^{|\beta_k - \beta_\ell|} = \frac{(\gcd(n_k, n_\ell))^{2\alpha}}{(n_k n_\ell)^{\alpha}}.$$

Lemma on GCD sums

Setting $c_j \equiv 1$ in the lemma on Poisson integrals, we get

Lemma

For arbitrary positive integers $n_1, n_2, ..., n_N$ written as $n_j = p^{\beta_j}$, we have

$$\sum_{k,\ell=1}^{N} \frac{(\gcd(n_k, n_\ell))^{2\alpha}}{(n_k n_\ell)^\alpha} = \int_{\mathbb{T}^d} \Big| \sum_{j=1}^{N} z^{\beta_j} \Big|^2 P_d(p^{-\alpha}, z) d\sigma_d(z).$$

We are particularly interested in estimating how large such sums can be when N is fixed and the positive integers are distinct; to this end, set

$$\Gamma_\alpha(N) = \sup \frac{1}{N} \sum_{k,\ell=1}^{N} \frac{(\gcd(n_k, n_\ell))^{2\alpha}}{(n_k n_\ell)^\alpha},$$

where the supremum is taken over all distinct $n_1, ..., n_N$.

Estimates of $\Gamma_\alpha(N)$

The study of such GCD sums was initiated by Koksma (1930s) who observed that they show up in uniform distribution theory. Erdős (1940) raised the problem of estimating $\Gamma_1(N)$; Gál (1949) solved this problem and showed that

$$\Gamma_1(N) \simeq (\log\log N)^2.$$

Dyer and Harman (1986), motivated by applications in metric Diophantine approximation, proved that

$$\Gamma_{1/2}(N) \ll \exp\left(\frac{c\log N}{\log\log N}\right)$$

and also that

$$\Gamma_\alpha(N) \ll \exp\left((\log N)^{(4-4\alpha)/(3-2\alpha)}\right)$$

for $1/2 < \alpha < 1$.

A sharp result for $0 < \alpha < 1$

Theorem (Aistleitner–Berkes–Seip 2013)

$$\Gamma_\alpha(N) \leq \begin{cases} \exp\left(c(\alpha)(\log N)^{1-\alpha}(\log\log N)^{-\alpha}\right), & 1/2 < \alpha < 1 \\ \exp\left(c(\log N \log\log N)^{1/2}\right), & \alpha = 1/2 \\ N^{(1+o(1))(1-2\alpha)}, & 0 < \alpha < 1/2 \end{cases}$$

when $N \to \infty$.

These estimates are best possible up to the precise value of $c(\alpha)$, with a possible exception for the case $\alpha = 1/2$.

About the proof of this theorem

Summarized in two sentences: The body of the proof is a *combination* of (1) a combinatorial reasoning (coming from Gál's work) regarding the left-hand side of the identity

$$\sum_{k,\ell=1}^{N} r^{|\beta_k-\beta_\ell|} c_k \overline{c}_\ell = \int_{\mathbb{T}^d} \Big| \sum_{j=1}^{N} c_j z^{\beta_j} \Big|^2 P_d(r,z)\, d\sigma_d(z).$$

when $c_k \equiv 1$ and (2) an analytic argument applied to the right-hand side of the same identity. The point of the combinatorial argument is to prepare for the analytic argument by reducing our consideration to multi-indices of small support (of order $\log N$).

By our lemma on Poisson integrals, we may as well view our theorem as a function theoretic result. In this perspective, I find the application of Gál's argument a surprising contribution of combinatorics to function theory in polydiscs.

Application: A Carleson–Hunt-type inequality

Theorem (Aistleitner–Berkes–Seip 2013)

For every 1*-periodic real-valued function* f *of mean 0 and bounded variation, there exists a constant* c *such that the following holds. For every finite and strictly increasing sequence of positive integers* $(n_k)_{1 \le k \le N}$ *and every associated finite sequence of real numbers* $(c_k)_{1 \le k \le N}$*, we have*

$$\int_0^1 \left(\max_{1 \le M \le N} \left| \sum_{k=1}^{M} c_k f(n_k x) \right| \right)^2 dx \le c \, (\log \log N)^4 \sum_{k=1}^{N} c_k^2.$$

Thus: The “price” we pay when replacing $\cos(2\pi x)$ (classical Carleson–Hunt) by a function f of bounded variation is a factor $(\log \log N)^4$. (Here the best possible power of $\log \log N$ is known to lie in the interval $[2, 4]$.)

Remark on the Riemann zeta-function

Observation: Our estimates for $\Gamma_\alpha(N)$ are of the same order of magnitude as the conjectured maximal order of $\zeta(\alpha + it)$ for $N < t < 2N$.

This looks as a mere coincidence and could be so too, but, on the other hand, a concrete link has been found in a closely related situation: Hilberdink (2009) has shown that the largest eigenvalue of the matrix

$$\left(\frac{(\gcd(n,m))^{2\alpha}}{(nm)^\alpha} \right)_{n,m=1}^N$$

can be used to estimate from below the maximum of $\zeta(\alpha + it)$ for $N < t < 2N$, but his bound is a little smaller than the best known lower bound for this maximum.