HÖRMANDER’S IMPACT ON PDE:S

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There are a few mathematicians in each generation who deserve to be called "great". One of them in the second half of the XXth century was

Lars Valter Hörmander (24 January 1931 – 25 November 2012)

a Swedish mathematician who has been called "the foremost contributor to the modern theory of linear partial differential equations". He was awarded the Fields Medal in 1962, the Wolf Prize in 1988, and the Leroy P. Steele Prize in 2006. His Analysis of Linear Partial Differential Operators, I –IV is considered a standard work on the subject of linear partial differential operators.

Hörmander completed his Ph.D. in 1955 at Lund University. He worked later at Stockholm University, at Stanford University, and at the Institute for Advanced Study in Princeton. He returned to Lund University as professor from 1968 until 1996, when he retired with the title of professor emeritus.
ON THE THEORY
OF GENERAL PARTIAL
DIFFERENTIAL OPERATORS

by
Lars Hörmander

in Lund

ACTA MATHEMATICA,

94 (1955), 161-248
A general representation for solutions of the equation

\[ P(D)u = 0 \]  \quad (1)

was constructed not so long before Hörmander’s work. It is a far-ranging generalization of the elementary representation of solutions to homogeneous ordinary linear differential equations with constant coefficients as the sum of exponential-power solutions. I recall that exponents in exponential functions are roots of the characteristic polynomial.
For partial differential equations the corresponding representation is

\[ u(x) = \int_{P(\xi)=0} e^{i(x, \xi)} \mu(d\xi), \quad (2) \]

where \( \mu \) is an arbitrary distribution from a certain class. In particular, \( \mu \) is the measure if the roots of \( P(\xi) \) are simple (L. Ehrenpreis, 1954).

Representation (2) follows from the fact that equation (1) considered in \( \mathbb{R}^n \) is equivalent to the equality

\[ P(\xi)(Fu)(\xi) = 0 \]

for distributions. This means that the distribution \( Fu := \mu \) does not vanish only on the set \( \{\xi : P(\xi) = 0\} \) which implies (2).
Formula (2) implies the existence of infinitely many solutions to an arbitrary homogeneous equation with constant coefficients. Hence it is natural to put a question: what are complementary conditions allowing to select a unique solution. We know that for equations of mathematical physics one considers boundary value problems, that is one chooses a solution satisfying prescribed boundary conditions.

A boundary value problem is called well-posed for a couple of function spaces \((A, B)\) if it is uniquely solvable in \(A\) for all data in \(B\) and the solution continuously depends on these data.
Question: Is there any well-posed problem for the equation $P(D)u = 0$ in a fixed domain?

Hörmander (Acta Mathematica, 1955) answered in affirmative if $A = B = L^2(\Omega)$, where $\Omega$ is a bounded domain. To explain his result, we need some notions.
Together with the operator $P(D)$, we consider the formally adjoint operator $\overline{P}(D)$ with coefficients complex-conjugate to the coefficients of $P(D)$. Given the operator $P$, one constructs the so-called minimal operator $P_0$ obtained as the closure in $L^2(\Omega)$ of the operator $P$ given on $C_0^\infty(\Omega)$. The operator adjoint to $\overline{P}_0$ in $L^2(\Omega)$ is called maximal.

Thus, the set of functions satisfying any homogeneous conditions contains the domain of the minimal operator and is contained in the domain of the maximal operator.

The question of existence of well-posed problems for $P(D)$ can be stated now as: is there an extension of the minimal operator which is a restriction of the maximal operator and possesses a bounded inverse on the whole of $L^2(\Omega)$?
It is appropriate to quote here Hörmander’s doctoral adviser Lars Gårding:

*When a problem of partial differential operators has been fitted into the abstract theory, all that remains is usually to prove a suitable inequality and much of our new knowledge is, in fact, essentially contained in such inequalities.*

Hörmander repeatedly followed this principle in his work. In particular, his necessary and sufficient condition for existence of...
the aforementioned extension requires that all functions $u \in C_0^\infty(\Omega)$ obey the inequality

$$\| u \|_{L^2(\Omega)} \leq C \| P(D) u \|_{L^2(\Omega)}$$

with $C$ independent of $u$.

For the Laplace operator and a bounded domain this estimate is a simple corollary of the Poincaré inequality between $\| u \|$ and $\| \text{grad} \ u \|$.

Hörmander’s result is a proof of the unexpected fact that such inequality holds for any differential operator with constant coefficients.
This fact is a particular case of general theorems due to Hörmander on the comparison of differential operators. By definition, the operator $P(D)$ is stronger than the operator $Q(D)$, if $D(P_0) \subset D(Q_0)$, where $P_0$ and $Q_0$ are corresponding minimal operators.

General arguments similar to the closed graph theorem show that $P$ is stronger than $Q$ if and only if all $u \in C_0^\infty(\Omega)$ obey the estimate

$$\|Q u\|_{L^2(\Omega)} \leq C \left( \|P u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right),$$

which is equivalent to

$$\|Q u\|_{L^2(\Omega)} \leq C_1 \|P u\|_{L^2(\Omega)}. \quad (4)$$
Developing a technique of energy integrals, Hörmander found the following algebraic necessary and sufficient condition for (4):

\[ |Q(\xi)| \leq C \sum |P^{(\alpha)}(\xi)|, \quad \forall \xi \in \mathbb{R}^n, \]

where \( P^{(\alpha)}(\xi) \) are derivatives of \( P(\xi) \) and the sum is taken over all multi-indeces \( \alpha \).

In particular, this implies that \( P(D) \) is an elliptic operator if and only if it is stronger than any operator of order not exceeding the order of \( P \).
Hörmander also showed that the operator $Q_0 P_0^{-1}$ is compact if and only if

$$\frac{\sum |Q^{(\alpha)}(\xi)|}{\sum |P^{(\alpha)}(\xi)|} \to 0 \quad \text{as} \quad \xi \to \infty.$$ 

Further, he proved that if $P$ and $Q$ are maximal operators and $D(P) \subset D(Q)$, then either $Q = aP + b$ with constant $a$ and $b$, or $P$ and $Q$ are ordinary differential operators such that the order of $Q$ does not exceed the order of $P$. 
Note that the question of existence of well-posed boundary value problems for systems is much more complicated than for a single equation. Even in the case of the simple system

\[ u_x + v_y = f_1, \quad v_x = f_2 \]

the minimal operator does not have solvable extensions. At present, a necessary and sufficient condition for existence of the well-posed problem from \( L^2(\Omega) \) to \( L^2(\Omega) \) is known for systems with constant coefficients.

Thus, the problem of existence of well-posed problems for operators with constant coefficients has been completely solved. However, explicit description of all well-posed boundary value problems for general operators is not yet given, although much is known for particular classes of equations.
Description of domains of maximal differential operators is a challenging open problem. An example of results which could be obtained is the following estimate of boundary traces of functions in $D(P)$ for $P(D_t, D_x)$ in the half-space $t > 0$ obtained by Gelman and M. in 1974:

$$
\| R(D)u \|_{H^\mu(\partial \mathbb{R}^n_+)}^2 \leq c \left( \| P(D)u \|_{L^2(\mathbb{R}^n_+)}^2 + \| u \|_{L^2(\mathbb{R}^n_+)}^2 \right).
$$

Let $H(\xi, \tau)$ be a polynomial in $\tau$ with roots in the half-plane $\text{Im} \, \zeta > 0$, $\zeta = \tau + i\sigma$, and such that

$$
|P(\xi, \tau)|^2 + 1 = |H(\xi, \tau)|^2.
$$

Then estimate (5) holds if and only if

$$
\int_{\mathbb{R}} \frac{|T_1(\xi, \tau)|^2 + |T_2(\xi, \tau)|^2}{|P(\xi, \tau)|^2 + 1} \, d\tau \leq c \left( 1 + |\xi|^2 \right)^{-\mu},
$$

where $T_1(\xi, \tau), T_2(\xi, \tau)$ denote the quotient and the remainder obtained when the polynomial (in $\tau$) $R(\xi, \tau)H(\xi, \tau)$ is divided by $P(\xi, \tau)$. 
Hypoelliptic operators with constant coefficients

It is well-known that any twice continuous differentiable solutions of Laplace’s equation is in fact an analytic function. An analogous property applies for any elliptic equations and systems with analytic coefficients. It was shown by Petrovskii that analyticity of all solutions is a characteristic property of operators of elliptic type with constant coefficients. Thus, a simple "superficial" property of the equation - ellipticity - turns out to be equivalent to a profound and considerably less obvious property - the analyticity of its solutions.

L. Schwartz posed the question of describing more general differential operators $P(D)$ having the property that any solution of the equation $P(D)u = f$ is in $C^\infty$ if $f \in C^\infty$. Such differential operators are called hypoelliptic. The simplest example of a differential operator which is hypoelliptic, but not elliptic, is the heat-conduction operator.
An exhaustive discussion of hypoelliptic operators with constant coefficients was given in 1955 by Hörmander. It turns out that for the operator $P$ to be hypoelliptic it is necessary and sufficient that for all zeros of the polynomial $P(\xi)$, $|\xi| \to \infty$ should imply $\text{Im} |\xi| \to \infty$.

If the polynomial $P(\xi)$ satisfies this condition, then there exists a positive constant $\gamma$ such that the inequality

$$|\text{Im} \xi| \geq a |\text{Re} \xi|^\gamma - b$$

is satisfied for the zeros of $P(\xi)$, where $a$ and $b$ are constants. The minimal value of $\gamma$ is called the exponent of hypoellipticity. It is always the case that $\gamma \leq 1$, where $\gamma = 1$ is equivalent to an elliptic operator. The index of the hypoellipticity can be defined as

$$\gamma = \lim \sup_{|\xi| \to \infty} \frac{\log(|\text{grad} P(\xi)|/|P(\xi)|)}{\log|\xi|}.$$
Hörmander collected his early results on partial differential equations with variable coefficients in the book

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1963
Hans Levy’s counterexample (1957). The equation
\[-i \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} - 2(x_1 + ix_2) \frac{\partial u}{\partial x_3} = f\]
with a certain \(f \in C^\infty\) has no solutions in \(D'\) for any domain in \(\mathbb{R}^3\).

A partial differential operator \(P(x, D)\) in \(\Omega \subset \mathbb{R}^n\) is locally solvable if every point \(x_0 \in \Omega\) has a neighborhood \(U \subset \Omega\) such that the equation

\[P(x, D)u = f\]

can be solved in \(D'\) for every \(f \in C^\infty_0(U)\).
Let $p$ denote the principal part of the symbol of $P(x, D)$. Then $P$ is said to be of principal type if

$$p(x, \xi) = 0, \quad \xi \in \mathbb{R}^n \setminus \{0\}, \quad \text{implies} \quad \nabla_\xi p(x, \xi) \neq 0. \quad (6)$$

By Hörmander (1955), in the constant coefficient case, (6) is necessary and sufficient for $P(D)$ to be stronger than any operator of lower order.

In his thesis Hörmander also showed that if the coefficients of $P$ are real then $P$ is locally solvable if it is of principal type.

In 1960 Hörmander found certain separately necessary and sufficient conditions for local solvability of differential operators with complex coefficients involving Levy’s example.
The next important result is due to Nirenbers and Treves (1970) who introduced the so-called \((\Psi)\)-condition: \(\text{Im}(ap)\) does not change sign from \(-\) to \(+\) along oriented bichacacteristics of \(\text{Re } p\). Here \(a\) is a function without zeros.

Nirenberg and Treves proved the equivalence of \((\Psi)\) and the local solvability for the first order operators with real analytic coefficients.

They conjectured that \((\Psi)\) is necessary and sufficient for the local solvability in the non-analytic case.
By duality, Nirenberg and Treves reduced the question of equivalence to the estimate

$$\|u\|_{L^2(U)} \leq C \|P^t u\|_{H^{1-m}(U)}$$

(7)

for small open sets $U$, where $m$ is the order of $P$ and $P^t$ is formally adjoint operator.

The necessity of $(\Psi)$ in the non-analytic case was justified by Hörmander in 1981.

N. Lerner constructed examples of operators showing that $(\Psi)$ is not sufficient for solvability with loss of one derivative.
N. Dencker (2004) proved that the condition \(\Psi\) is sufficient for the local solvability with loss of two derivatives. Instead of (7) he achieved the inequality

\[
\|u\|_{L^2(U)} \leq C \left( \|P^t u\|_{H^{2-m}(U)} + \text{weak terms} \right).
\]

Later he replaced 2 by \(3/2 + \varepsilon\).

Finally, in 2006 N. Lerner obtained the best result removing \(\varepsilon\).
In 1966 Hörmander published a paper in "Annals of Mathematics" on a general theory for non-elliptic problems. He reduced the BVP to a boundary pseudo-differential system which is not necessarily elliptic. The motivation came from the so-called \( \overline{\partial} \)-Neumann problem recently addressed by Kohn, Morrey et al.
Hörmander’s purpose was to give a complete study of the case when

\[ Pu \in H^s_{\text{loc}} \quad \text{implies} \quad u \in H^{s+m-1/2}_{\text{loc}} \]

The term subelliptic operator was introduced by Hörmander for what is now called a subelliptic operator with loss of \( \frac{1}{2} \) derivatives. Such operators were characterized by the condition \( \{ \text{Re} \ p, \text{Im} \ p \} > 0 \) at the zeros of the principal symbol, where \( \{ f, g \} \) is the Poisson bracket, that is

\[ \{ f, g \} = \sum (\partial f / \partial \xi_j \partial g / \partial x_j - \partial f / \partial x_j \partial g / \partial \xi_j). \]
The complete characterization of the loss of smoothness $\delta \in (0, 1)$ required work of Hörmander, Nirenberg-Treves, Egorov, Fefferman, Dencker, Lerner. The main result for the scalar operator is as follows.

**Theorem.** The pseudo-differential operator $P$ with principal symbol $p$ is subelliptic at $\gamma_0 \in T^*(X) \setminus 0$ with loss of $\delta < 1$ derivatives if and only if there is a neighborhood $V$ of $\gamma_0$ such that

(i) For every $\gamma \in V$ the repeated Poisson bracket

$$(H_{\text{Re}} z p)^j \text{Im } z p(\gamma)$$

is different from 0 for some $z \in \mathbb{C}$ and some $j \leq \delta / (1 - \delta)$. (ii) (8) is non-negative of $j$ is odd and $j$ is the smallest integer such that (8) does not vanish for all $z \in \mathbb{C}$. 


The important so called sharp Gårding inequality appeared in the "Annals" (1966) paper by Hörmander.

If $a \in S^{2m+1}$ and $\text{Re } a \geq 0$, then

$$\text{Re} \left( a(x, D)u, u \right) \geq -C \|u\|_{H^m}^2, \quad u \in C_0^\infty.$$ 

For the matrix case see Lax and Nirenberg, C.P.A.M. (1966).

For the proofs in the papers just mentioned it is essential that the symbol is sufficiently smooth. I quote my result of 1973 which shows how one can weaken this restriction and make the sharp Gårding inequality more precise.

**Theorem.** Let

\[ \sigma(x, \xi) = |f(x)| e(x, \xi) |\xi|, \]

where \( e \) is a sufficiently smooth symbol of order zero, \( e(x, \xi) > 0 \) for \( |\xi| \neq 0 \), and let \( f \) be a function in \( C^3 \) such that \( x_n f(x) \geq 0 \). Then the inequality

\[ \text{Re} (\sigma u, u) \geq c_1 \| |f|^{1/2} e^{1/2} u \|_{H^{1/2}}^2 - c_2 \| u \|_{L^2}^2 \]

holds for all \( u \in C_0^\infty \). Here \( e^{1/2} \) is a singular integral operator with the symbol \( e^{1/2}(x, \xi) \).
Hypoelliptic operators with variable coefficients

I recall that \( P(x, D) \) is called hypoelliptic if

\[
\text{sing supp } u = \text{sing supp } P(D)u, \quad u \in \mathcal{D}'.
\]

For operators with constant coefficients this is equivalent to microhypoellipticity

\[
WF(u) = WF(P(D)u), \quad u \in \mathcal{D}'.
\]

(For a distribution \( u \in \mathcal{D}'(X) \) on a \( C^\infty \) manifold \( X \) Hörmander, 1970, defines a set

\[
WF(u) \subset T^*(X) \setminus 0
\]

with projection in \( X \) equal to \( \text{sing supp } u \), which is conic with respect to multiplication by positive scalars in the fibers of the cotangent bundle \( T^*(X) \). He calls it the \textit{wave front set} of \( u \) by analogy with the classical Huyghens construction of a propagating wave.)
In more familiar terms, $WF(f)$ tells not only where the function $f$ is singular (which is already described by its singular support) but also how and why it is singular, by being more exact about the direction in which the singularity occurs.

Microhypoellipticity implies hypoellipticity but the converse is not always true.

In 1955 Hörmander proved that the hypoellipticity of $P(D)$ is equivalent to the condition: for some $\gamma > 0$

$$P^{(\alpha)}(\xi)/P(\xi) = O(|\xi|^{-\gamma|\alpha|}) \quad \text{for every } \alpha \text{ when } \xi \to \infty \text{ in } \mathbb{R}^n.$$
He also verified the hypoellipticity of operators $P(x, D)$ of constant strength, that is differential operators $P(x, D)$ such that

$$|D_x^\alpha D_\xi^\beta P(x, \xi)| \leq C_{\alpha\beta} |P(x, \xi)| |\xi|^{-\gamma|\alpha|}$$

for large $|\xi|$. This condition is not invariant under coordinate changes. It does not cover some types of hypoelliptic operators occurring in probability theory, such as, for instance, the Kolmogorov equation

$$\partial^2 u/\partial x^2 + x \partial u/\partial y - \partial u/\partial t = f,$$

which describes Brownian motion on $\mathbb{R}$, $x$ denoting the velocity and $y$ the position of a particle.
Hörmander’s paper of 1967 in ”Acta Math.” is devoted to a study of ”generalized Kolmogorov operators” of the form

\[ \sum X_j^2 + X_0, \]

where \( X_j \) are real vector fields. It is proved that such operators are hypoelliptic if the Lie algebra generated by all \( X_j \) gives a basis for all vector field at any point.

An active role in this development was played by Melin, Sjöstrand, Oleinik & Radkevich.
Lars Hörmander

THE ANALYSIS OF LINEAR
PARTIAL DIFFERENTIAL OPERATORS

Volumes I - IV

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1983
Chapter I. Test Functions
Chapter II. Definition and Basic Properties of Distributions
Chapter III. Differentiation and Multiplication by Functions
Chapter IV. Convolution
Chapter V. Distribution in Product Spaces
Chapter VI. Composition with Smooth Maps
Chapter VII. The Fourier Transform
Chapter VIII. Spectral Analysis of Singularities
Chapter IX. Hyperfunctions
Chapter X. Existence and Approximation of Solutions of Differential Equations

Chapter XI. Interior Regularity of Solutions of Differential Equations

Chapter XII. The Cauchy and Mixed Problems

Chapter XIII. Differential Operators of Constant Strength

Chapter XIV. Scattering Theory

Chapter XV. Analytic Function Theory and Differential Equations

Chapter XVI. Convolution Equations
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Chapter XIX. Elliptic Operators on a Compact Manifold Without Boundary

Chapter XX. Boundary Problems for Elliptic Differential Operators

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Chapter XXIII. The Strictly Hyperbolic Cauchy Problem

Chapter XXIV. The Mixed Dirichlet-Cauchy Problem for Second Order Operators
Chapter XXV. Lagrangian Distributions and Fourier Integral
Chapter XXVI. Pseudo-Differential Operators of Principal Type
Chapter XXVII. Subelliptic Operators
Chapter XXVIII. Uniqueness for the Cauchy Problem
Chapter XXIX. Spectral Asymptotics
Chapter XXX. Long Range Scattering Theory
Hörmander’s results on non-elliptic problems have direct applications to the degenerate oblique derivative problem. Let \( \Omega \subset \mathbb{R}^n, n \geq 3 \), be a bounded domain with smooth boundary \( \Gamma \), and denote the exterior unit normal to \( \Gamma \) by \( \nu \). The oblique derivative or Poincaré problem consists in determining a function \( u \) satisfying
\[
\Delta u = 0 \quad \text{in} \ \Omega, \quad \partial u / \partial \ell = f \quad \text{on} \ \Gamma,
\]
where \( \ell \) denotes a field of unit vectors on \( \Gamma \). The problem (9) can be converted into a pseudodifferential equation of first order on \( \Gamma \) with the principal symbol
\[
\sigma_0(x, \xi) = -\cos(\nu, \ell)|\xi| + i \cos(\xi, \ell)|\xi|, \ x \in \Gamma, \ \xi \in T_x \Gamma,
\]
where \( T_x \) stands for the tangent space at the point \( x \). Observe that this equation is elliptic if and only if the vector field \( \ell \) is nowhere tangent to \( \Gamma \).
In the elliptic case, the Fredholm property and regularity of problem (9) follow from standard elliptic theory of pseudodifferential operators, while its unique solvability is a consequence of Giraud’s theorem on the sign of the oblique derivative at the extremum point.

Until the mid-sixties almost nothing was known about the degenerate problem (9). For *transversal degeneration* where the field $\ell$ is tangent to $\Gamma$ on some $(n - 2)$–dimensional submanifold $\Gamma_0$, but is not tangential to $\Gamma_0$, this situation changed when the first results on non–elliptic pseudodifferential operators became available.
As a by–product of his subelliptic estimates for pseudodifferential equations, Hörmander proved that the dimension of the kernel of this problem may be infinite or the regularity of solutions may fail.

Geometrically, the transversal degeneration leads to the following three types of components of the set $\Gamma_0$ (where the vector field $\ell$ is tangent to $\Gamma$): those consisting of the so–called ”entrance” points (of $\ell$ into $\Omega$), ”exit” points, and ”status quo” points where $\ell$ remains on the same side of $\Gamma$; see Figs. 1 – 3.
In 1969, after the Malyutov, Egorov & Kondrat'ev and Maz’ya & Paneyah studies, the following properties of transversal degeneration became clear.

The *status quo* components do not affect the unique solvability of the problem; they only generate some loss of regularity of solutions. In order to preserve unique solvability, one should allow discontinuities of solutions on the entrance components and prescribe additional boundary conditions on the exit components.
Around 1970 V. Arnold stressed the importance of the so-called *generic case of degeneration*, where the vector field $\ell$ is no longer transversal to $\Gamma_0$. More precisely, one assumes that there are smooth manifolds (without boundary) $\Gamma_0 \supset \Gamma_1 \ldots \supset \Gamma_s$ of dimensions $n - 2, n - 3, \ldots, n - 2 - s$ such that $\ell$ is tangent to $\Gamma_j$ exactly at the points of $\Gamma_{j+1}$, whereas $\ell$ is nowhere tangent to $\Gamma_s$; see Fig. 4. A local model of this situation is given by the following:

$$
\Omega = \{ x \in \mathbb{R}^n : x_1 > 0 \}, \quad \Gamma = \{ x_1 = 0 \},
$$

$$
\ell = x_2 \partial_1 + x_3 \partial_2 + \ldots + x_k \partial_{k-1} + \partial_k, \quad k \leq n,
$$

$$
\Gamma_j = \{ x_1 = x_2 = \ldots = x_{2+j} = 0 \}, \quad j = 0, \ldots, k - 2.
$$
The generic case is much more difficult from the analytical point of view than the transversal one, because entrance and exit points are permitted to belong to one and the same component of \( \Gamma_0 \) and the usual localization technique does not apply.

Even a question of hypoellipticity is still open for the generic case. It could be asked as follows:

Denote by \( u \) a solution of the oblique derivative problem

\[
\Delta u = 0 \quad \text{in} \quad \mathbb{R}_+^3, \\
\frac{\partial u}{\partial x_1} + x_1 \frac{\partial u}{\partial x_2} + x_2 \frac{\partial u}{\partial x_3} = \varphi \quad \text{for} \quad x_3 = 0.
\]

Let \( \varphi \in C^\infty(\mathbb{R}^2) \) and \( u \in C^\infty(\mathbb{R}^2 \setminus 0) \). Is it true that \( u \in C^\infty(\mathbb{R}^2) \)?
In 1972, I published a result related to the generic degeneration, still the only known one. It turned out that the manifolds $\Gamma_j$ of codimension greater than one do not influence the correct statement of the problem, contrary to Arnold’s expectations. By the way, a description of the asymptotics of solutions near the points of tangency of the field $\ell$ to $\Gamma_0$ remains a difficult long-standing problem.
**Fig. 1** Entrance points – under-determined problem

**Fig. 2** Exit points – over-determined problem
Fig. 3  Status quo – well-posed problem

Fig. 4  Generic case of degeneration
THANK YOU!