Heisenberg's uncertainty principle in the sense of Beurling

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Heisenberg's uncertainty principle

Heisenberg's uncertainty principle asserts that a function $f \in L^2(\mathbb{R})$ and its Fourier transform

$$\hat{f}(y) := \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi yx} f(x) \mathrm{d}x$$

cannot both be too localized. In quantum theory, this corresponds to our inability to measure the position and the momentum simultaneously. a precise version asserts that

$$\|Mf\|_{L^2(\mathbb{R})} \|M\hat{f}\|_{L^2(\mathbb{R})} \geq \frac{1}{4\pi} \|f\|^2_{L^2(\mathbb{R})},$$

where we write $Mf(x) := xf(x)$ (multiplication by the independent variable).

Beurling's version

Building on work of Hardy, Beurling obtained a version which is attractive for its simplicity and beauty.

Theorem (Beurling)

If $f \in L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{2\pi|xy|} |f(x)\hat{f}(y)| \mathrm{d}x \mathrm{d}y < +\infty,$$

then $f = 0$ a.e.

Remark

The constant 2π in the exponent is optimal.

Historical comments

The above theorem has an interesting history. The result was found among Beurling's unpublished material, and appeared without proof in the 1989 Collected Works volume 2. Subsequently, it transpired that Hörmander had retained a copy of Beurling's original proof. In [Ark. Mat. 29 (1991), 237-240], Hörmander writes "The editors state that no proof has been preserved. However, in my files I have notes which I made when Arne Beurling explained this result to me during a private conversation [in] the years 1964-1968 when we were colleagues at IAS [in Princeton]".

Remarks on Beurling's approach

We remark that Beurling's proof is based on the idea that the functions

$$\mathcal{M}_f(y) := \int_{\mathbb{R}} \mathrm{e}^{2\pi|xy|}|f(x)|\mathrm{d}x, \quad \mathcal{M}_{\hat{f}}(x) := \int_{\mathbb{R}} \mathrm{e}^{2\pi|xy|}|\hat{f}(y)|\mathrm{d}y$$

are integrability weights for $\hat{f}$ and f, respectively:

$$\int_{\mathbb{R}} \mathcal{M}_f(y)|\hat{f}(y)|\mathrm{d}y = \int_{\mathbb{R}} \mathcal{M}_{\hat{f}}(y)|f(x)|\mathrm{d}x < +\infty.$$

Here, we present a new and completely different proof, published in [J. Anal. Math. 118 (2012), 691-702].

A new approach

We consider the function

$$F(\lambda) := \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi\lambda xy} \bar{f}(x) \hat{f}(y) \mathrm{d}x \mathrm{d}y. \tag{1}$$

Here, λ is initially real, but might also be complex if the convergence permits that. Let $\mathcal{S}$ denote the open strip

$$\mathcal{S} := \{\lambda \in \mathbb{C} : -1 < \mathrm{Im}\,\lambda < 1\}$$

and $\overline{\mathcal{S}}$ its closure. By the triangle inequality for integrals,

$$|F(\lambda)| \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{-2\pi xy \mathrm{Im}\,\lambda} |f(x) \hat{f}(y)| \mathrm{d}x \mathrm{d}y,$$

so that $F(\lambda)$ is well-defined for all $\lambda \in \overline{\mathcal{S}}$ in the setting of Beurling's theorem. Moreover, by an argument involving uniform convergence, $F(\lambda)$ is continuous and bounded in $\overline{\mathcal{S}}$, and holomorphic in $\mathcal{S}$.

Statement of the main theorem

We first observe that in Beurling's theorem, we have from the inequality $\mathrm{e}^{2\pi|xy|} \geq 1$ that

$$\|f\|_{L^1(\mathbb{R})}\|\hat{f}\|_{L^1(\mathbb{R})} \leq \int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{e}^{2\pi|xy|}|f(x)\hat{f}(y)|\mathrm{d}x\mathrm{d}y < +\infty,$$

which means that we automatically have that $f, \hat{f} \in L^1(\mathbb{R})$. In particular, $f \in L^p(\mathbb{R})$ for all p with $1 \leq p \leq +\infty$.

Theorem

Suppose $f \in L^2(\mathbb{R})$ and that the function $F(\lambda)$ given by (1) – initially for real λ – extends holomorphically to a neighborhood of $\bar{\mathbb{D}} \setminus \{\pm\mathrm{i}\}$. If, in addition, $F(\lambda)$ is bounded in $\mathbb{D}$, then $f = 0$ a.e.

Remark

Since $\bar{\mathbb{D}} \setminus \{\pm\mathrm{i}\}$ is contained in the open strip $\mathcal{S}$, the assumptions are now rather substantially weaker than in Beurling's theorem.

Fourier inversion and symmetry

If λ is real and nonzero ($\lambda \in \mathbb{R}^\times$), Fourier inversion tells us that

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi\lambda xy} \hat{f}(y)\mathrm{d}y = f(\lambda x)$$

holds for a.e. x. It follows that for such λ,

$$F(\lambda) = \int_{\mathbb{R}} \bar{f}(x) f(\lambda x)\mathrm{d}x.$$

By a change-of-variables, we see that

$$F(\lambda) = \int_{\mathbb{R}} \bar{f}(x) f(\lambda x)\mathrm{d}x = \frac{1}{|\lambda|} \int_{\mathbb{R}} \bar{f}(y/\lambda) f(y)\mathrm{d}y, \qquad \lambda \in \mathbb{R}^\times,$$

so that

$$F(\lambda) = \frac{1}{|\lambda|} \bar{F}(1/\lambda), \qquad \lambda \in \mathbb{R}^\times. \tag{2}$$

The proof: preliminary observations

The assumption that $f \in L^2(\mathbb{R})$ shows that $F(\lambda)$ is continuous in $\lambda \in \mathbb{R}^\times$. Also, we note that

$$F(1) = \int_{\mathbb{R}} \bar{f}(x) f(x) \mathrm{d}x = \int_{\mathbb{R}} |f(x)|^2 \mathrm{d}x = \|f\|^2_{L^2(\mathbb{R})},$$

so that if we can show that $F(\lambda)$ vanishes identically, then in particular $F(1) = 0$ so that $f = 0$ a.e. follows. Next, we put

$$\Phi(\lambda) := F(\lambda)\sqrt{1+\lambda^2}.$$

By the symmetry property (2), we have, for $\lambda \in \mathbb{R}^\times$,

$$\Phi(\lambda) = \frac{\sqrt{1+\lambda^2}}{|\lambda|} \bar{F}(1/\lambda) = \sqrt{1+\lambda^{-2}}\,\bar{F}(1/\lambda) = \bar{\Phi}(1/\lambda). \qquad (3)$$

The proof: the main argument

By assumption, $F(\lambda)$ has a (bounded) holomorphic extension to a neighborhood of $\bar{\mathbb{D}} \setminus \{\pm \mathrm{i}\}$, so that $\Phi(\lambda)$ does, too. If we define

$$\Phi_*(\lambda) := \bar{\Phi}(1/\bar{\lambda}),$$

the function $\Phi_*(\lambda)$ is holomorphic in a neighborhood of $\mathbb{C} \setminus (\mathbb{D} \cup \{\pm \mathrm{i}\})$, and is well-defined on $\mathbb{R}^\times$ as well. In view of (3), On $\mathbb{R}^\times$, $\Phi(\lambda) = \Phi_*(\lambda)$, so that $\Phi(\lambda)$ extends to a bounded holomorphic function in $\mathbb{C} \setminus \{\pm \mathrm{i}\}$. Note that no branching can take place! The singularities at $\pm \mathrm{i}$ are removable, so Liouville's theorem tells us that $\Phi(\lambda) \equiv \mathrm{const}$. Finally, since $F(\lambda)$ is bounded in $\mathbb{D}$,

$$\Phi(\lambda) = \sqrt{1 + \lambda^2} F(\lambda) \to 0 \quad \text{as} \quad \lambda \to \pm \mathrm{i},$$

so $\Phi(\lambda) \equiv 0$. It is now immediate that $F(\lambda) \equiv 0$, as needed!

An example showing sharpness of main theorem

Let us consider the function $f(x) = \mathrm{e}^{-\pi\beta x^2}$, where β is complex with positive real part. Then $f \in L^2(\mathbb{R})$, and the associated function $F(\lambda)$ is

$$F(\lambda) = \int_{\mathbb{R}} \bar{f}(x) f(\lambda x) \mathrm{d}x = \bar{\beta}^{-1/2} \left(1 + \frac{\beta}{\bar{\beta}} \lambda^2 \right)^{-1/2}.$$

This function $F(\lambda)$ is holomorphic in $\mathbb{D}$ but posesses two square root branch points at the roots of $\lambda^2 = -\bar{\beta}/\beta$. These roots lie on the unit circle $|\lambda| = 1$. This shows the importance of requiring analytic extension across every point of $\mathbb{T} \setminus \{\pm \mathrm{i}\}$.

A variation on Beurling's theme

If we believe the main enemy in Beurling's theorem is the Gaussian $f(x) = \mathrm{e}^{-\pi x^2}$, for which $\hat{f}(x) = \mathrm{e}^{-\pi x^2}$, we might try to replace the double integral by a single integral. More precisely, if $f, \hat{f} \in L^1(\mathbb{R})$ and if

$$\int_{\mathbb{R}} |f(x)\hat{f}(x)| \mathrm{e}^{2\pi x^2} \mathrm{d}x < +\infty,$$

does it follow that $f = 0$ a.e.? Actually, this is easy to defeat! The reason is that we can achieve disjoint supports of f and $\hat{f}$, so that the above integral may even vanish for nontrivial f. Here, I should thank M. Benedicks for help with the construction.