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  acts faithfully on $L_X = \pi_2(BT) \cong (\mathbb{Z}_p)^r$. ($X$ conn.)

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- $H^*(BX;\mathbb{Z}_p)\otimes\mathbb{Q} \simeq H^*(BT;\mathbb{Z}_p)^{W_X}\otimes\mathbb{Q} \cong \mathbb{Q}_p[x_{2d_1},\ldots,x_{2d_r}]$
  and hence $W_X \leq \mathrm{Aut}(L_X)$ is a finite $\mathbb{Z}_p$-reflection group.

Irred. finite $\mathbb{Q}_p$-reflection groups (Shephard-Todd, Clark-Ewing)

W	Order	Degrees	$\mathbb{Q}(\chi)$	Primes
Σ_{n+1} (family 1)	$(n+1)!$	$2, 3, \ldots, n+1$	$\mathbb{Q}$	all p
$G(m, s, n)$ (family 2a) $m \geq 2, n \geq 2, m \neq s$ if $n = 2$	$n! m^{n-1} \frac{m}{s}$	$m, 2m, \ldots, (n-1)m, n\frac{m}{s}$	$\mathbb{Q}(\zeta_m)$	$p \equiv 1 \quad (m)$; all p for $m = 2$
$D_{2m} = G(m, m, 2)$ (family 2b) $m \geq 3$	$2m$	$2, m$	$\mathbb{Q}(\zeta_m + \zeta_m^{-1})$	$p \equiv \pm 1 \quad (m)$; all p for $m = 3, 4, 6$
$C_m = G(m, 1, 1)$ (family 3) $m \geq 2$	m	m	$\mathbb{Q}(\zeta_m)$	$p \equiv 1 \quad (m)$; all p for $m = 2$
G_4	24	4, 6	$\mathbb{Q}(\zeta_3)$	$p \equiv 1 \quad (3)$
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G_{28}	1152	2, 6, 8, 12	$\mathbb{Q}$	all p
G_{29}	7680	4, 8, 12, 20	$\mathbb{Q}(\zeta_4)$	$p \equiv 1 \quad (4)$
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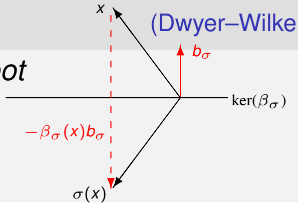
Root data for p-compact groups

(Dwyer–Wilkerson, A–G)

- *Every p–compact grp has a root datum $\mathbf{D}_X = (W_X, L_X, \{\mathbb{Z}_p b_\sigma\})$*
- *Classification of $\mathbb{Z}_p$–root data.*



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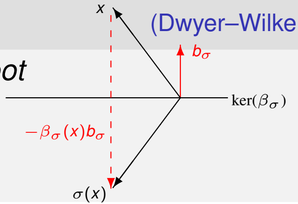

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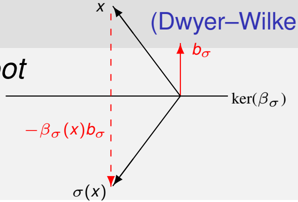

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Corollary

Every conn. 2–compact group BX can be written

$$BX \simeq BG^\wedge_2 \times B\,\mathrm{DI}(4)^s$$

G cpt Lie. $B\,\mathrm{DI}(4)$ = 2–compact grp corresponding to $(G_{24}, p = 2)$

Finite loop space = pointed space BX s.t. $H^*(\Omega BX;\mathbb{Z})$ f.g. over $\mathbb{Z}$.

Classification Theorem for finite loop spaces (A–G)

$$\{\textit{conn. ft. loop spaces}\} \twoheadrightarrow \{\ \{\mathbf{D}_p\}_p \mid deg(\mathbf{D}_p) = deg(\mathbf{D}_q)\ \}$$

$$Y \mapsto \{\mathbf{D}_{Y_{\hat{p}}}\}_p.$$

Preimage: $\mathrm{Out}(K_{\mathbb{Q}})\backslash \mathrm{Out}^c(K_{\mathbb{A}_f})/\prod_p \mathrm{Out}(\mathbf{D}_p)$

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Corollary ("Maximal torus conjecture") (A–G)

$$\{\textit{cpt. Lie grps}\} \overset{1-1}{\longleftrightarrow} \{\textit{ft. loop spaces w. max torus}\}$$

via $G \mapsto BG$.

If G conn: $\mathrm{Out}(BG) \cong \mathrm{Out}(G) \cong \mathrm{Out}(\mathbf{D}_G)$.

Theorem (Steenrod's problem for $\mathbb{F}_2$) (A–G)

If $H^*(X;\mathbb{F}_p)$ is a finitely generated polynomial algebra over $\mathbb{F}_p$, with generators in deg. ≥ 3, for some space X,
then $H^*(X;\mathbb{F}_p)$ is isomorphic, as a graded algebra, to a tensor product of copies of

$$
\begin{aligned}
H^*(B\,\mathrm{SU}(n);\mathbb{F}_2) &\cong \mathbb{F}_2[x_4, x_6, \ldots, x_{2n}],\\
H^*(B\,\mathrm{Sp}(n);\mathbb{F}_2) &\cong \mathbb{F}_2[x_4, x_8, \ldots, x_{4n}],\\
H^*(B\,\mathrm{Spin}(7);\mathbb{F}_2) &\cong \mathbb{F}_2[x_4, x_6, x_7, x_8],\\
H^*(B\,\mathrm{Spin}(8);\mathbb{F}_2) &\cong \mathbb{F}_2[x_4, x_6, x_7, x_8, y_8],\\
H^*(B\,\mathrm{Spin}(9);\mathbb{F}_2) &\cong \mathbb{F}_2[x_4, x_6, x_7, x_8, x_{16}],\\
H^*(BG_2;\mathbb{F}_2) &\cong \mathbb{F}_2[x_4, x_6, x_7],\\
H^*(BF_4;\mathbb{F}_2) &\cong \mathbb{F}_2[x_4, x_6, x_7, x_{16}, x_{24}],\\
H^*(B\,\mathrm{DI}(4);\mathbb{F}_2) &\cong \mathbb{F}_2[x_8, x_{12}, x_{14}, x_{15}].
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Theorem (Steenrod's problem for $\mathbb{Z}$) (A–G)

*If $H^*(X;\mathbb{Z})$ is a finitely generated polynomial algebra over $\mathbb{Z}$, for some space X,
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$$H^*(B\,\mathrm{SU}(n);\mathbb{Z}) \cong \mathbb{Z}[x_4, x_6, \dots, x_{2n}],$$
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$$H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[x_2].$$