Dimension splitting for quasilinear parabolic equations

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In the current paper, we derive a rigorous convergence analysis for a broad range of splitting schemes applied to abstract nonlinear evolution equations, including the Lie and Peaceman–Rachford splittings. The analysis is in particular applicable to (possibly degenerate) quasilinear parabolic problems and their dimension splittings. The abstract framework is based on the theory of maximal dissipative operators, and we both give a summary of the used theory and some extensions of the classical results. The derived convergence results are illustrated by numerical experiments.

Keywords: quasilinear parabolic problems, dimension splitting, convergence, degeneracy.

1. Introduction

A commonly applied class of time stepping schemes for nonlinear evolution equations of the form

\[ \dot{u} = (f_a + f_b)(u) \]

are the so called splitting methods. The concept behind these schemes is to approximate the semiflows generated by \( f_a \) and \( f_b \) separately in every time step. This strategy often results in a drastic reduction of the computational effort, when compared with methods based on the full vector field \( f_a + f_b \). See McLachlan & Quispel (2002); Hunsdorfer & Verwer (2003), for an introductory survey.

There has been a large number of studies dealing with splitting methods and nonlinear evolution equations, and we give a brief résumé of the major trends in the literature. Convergence results have been established for both general families of splitting schemes and abstract equations, see for example Brézis & Pazy (1970, 1972); Lions & Mercier (1979); Miyadera & Oharu (1970). Most of these results are connected to the development of the nonlinear semigroup theory. The drawback of such abstract results is that the related hypotheses are rather challenging to prove for a specific application. To overcome these disadvantages, many authors have restricted their attention to specific problem classes. Some examples where this has been achieved are dimension splittings for (scalar) conservation laws, Coron (1982); Crandall & Majda (1980); Teng (1994), hyperbolic parabolic splittings, Karlsen & Risebro (1997, 2000), splitting in connection with obstacle problems, Lions & Mercier (1979), and source term splittings, Faou (2007); Lubich (2008).

When considering the applications found in the literature it is striking that so little has been done for the dimension splitting of parabolic equations, which was the original motivation for introducing the classical Peaceman–Rachford splitting, Peaceman & Rachford (1955). One of the few exceptions is the numerical study by Hunsdorfer & Verwer (1989). The aim of this paper is therefore to establish a rigorous convergence analysis for a broad family of splitting methods applied to nonlinear parabolic problems.

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equations. We will especially consider parabolic problems governed by the quasilinear vector field
\[
fa + fb = \sum_{i=1}^t Di a_i(x, Du) + c_i Du + g_i(x) + \sum_{j=t+1}^d Dj a_j(x, Du) + c_j Du + g_j(x).
\] (1.1)

The paper is organized as follows. In Section 2, an abstract framework is presented based on maximal dissipative operators. Employing the convergence results of Brézis & Pazy (1972) and an extension of the ideas in Lions & Mercier (1979) enables us in Section 3 to derive a widely applicable convergence analysis, which includes the Lie and Peaceman–Rachford splittings as well as the “sum” splitting proposed by Coron (1982). In Section 4, we prove that parabolic equations with (nondegenerate) vector fields of the form (1.1) fit into the derived abstract framework. A further extension to the degenerate case is discussed in Section 5. We conclude by illustrating the derived convergence results by a set of numerical experiments in Section 6.

2. Abstract framework

Consider the nonlinear evolution equation
\[
\dot{u} = f(u) = (fa + fb)(u), \quad u(0) = u_0,
\] (2.1)
given on an arbitrary (real) Hilbert space \(H\). The norm and inner product of \(H\) are referred to as \(\| \cdot \|\) and \((\cdot, \cdot)\), respectively. We will distinguish between the sum operator \(fa + fb\): \(D(fa) \cap D(fb) \to H\) and the full operator \(f: D(f) \to H\), where the latter is a possible extension of the first, i.e.,
\[
fa + fb \subseteq f.
\]

A suitable setting, which includes parabolic problems, is to assume the following.

ASSUMPTION 2.1 The operators \(f, fa, fb\) are all densely defined and maximal dissipative on \(H\).

Recall that an operator \(g\) is said to be maximal dissipative on \(H\) if and only if \((g(u) - g(v), u - v) \leq 0\) for all \(u, v \in D(g)\), and \(I - \lambda g\) is onto for every \(\lambda > 0\). Note that no assumption is made regarding the maximality of \(fa + fb\).

PROPOSITION 2.1 A densely defined and maximal dissipative operator \(g\) generates a nonexpansive semigroup \(e^{\lambda t}\), and \(u(t) = e^{\lambda t}u_0\) is the (generalized) solution of \(\dot{u} = g(u), u(0) = u_0 \in H\).

This is one of the main results from the nonlinear semigroup theory, see for example (Zeidler, 1990, Corollary 31.1). Further discussion and extensions of the theory can be found in Barbu (1976); Crandall (1986). In preparation for the analysis, we collect some features of the resolvent and the Yosida approximation which are defined as
\[
R_{\lambda g} = (I - \lambda g)^{-1} \quad \text{and} \quad Y_{\lambda g} = \frac{1}{\lambda}(R_{\lambda g} - I) = gR_{\lambda g},
\]
respectively, for every densely defined maximal dissipative operator \(g\) and all \(\lambda > 0\). The following lemma is standard (Deimling, 1985, Proposition 11.3), but we give a short proof for the sake of completeness.

LEMMA 2.1 The operators \(R_{\lambda g}\) and \(Y_{\lambda g}\) satisfy the assertions below.

(a) \(R_{\lambda g}\) is nonexpansive.
Proof. The first assertion is a consequence of the maximal dissipativity as
\[ \| (I - \lambda g)(u) - (I - \lambda g)(v) \|^2 = \| u - v \|^2 - 2\lambda (g(u) - g(v), u - v) + \lambda^2 \| g(u) - g(v) \|^2 \geq \| u - v \|^2, \]
for all \( u, v \in D(g) \). The second assertion holds true for all \( u \in D(g) \), by the inequality
\[ \| R_{\lambda g}(u) - u \| = \| R_{\lambda g}(u) - R_{\lambda g}(I - \lambda g)(u) \| \leq \lambda \| g(u) \|. \]

The extension to all elements in \( H \) follows as \( D(g) \) is dense in \( H \) and \( R_{\lambda g} \) is Lipschitz continuous. To prove the third assertion, let \( u \in D(g) \) and observe that
\[ \| Y_{\lambda g}(u) \|^2 - (g(u), Y_{\lambda g}(u)) = \frac{1}{\lambda} (gR_{\lambda g}(u) - g(u), R_{\lambda g}(u) - u) \leq 0, \]
i.e., \( \| Y_{\lambda g}(u) \| \leq \| g(u) \| \). Next, assume that the subsequence \( \{ Y_{\lambda g}(u) \} \) is weakly convergent to \( y \in H \). Then
\[ 0 \geq \lim_{\lambda \to 0} (g(v) - Y_{\lambda g}(u), v - R_{\lambda g}(u)) = (g(v) - y, v - u), \]
for all \( v \in D(g) \). As \( I - g \) is onto, we can chose \( v \) such that \((I - g)(v) = u - y\), which implies that \( \| v - u \| \leq 0 \), i.e., \( v = u \) and \( y = g(u) \). This line of argumentation is valid for every weakly convergent subsequence of the (bounded) sequence \( \{ Y_{\lambda g}(u) \} \), hence, \( Y_{\lambda g}(u) \to g(u) \) when \( \lambda \) tends to zero (Gajewski et al., 1974, Lemma I.5.4). By these considerations,
\[ \| Y_{\lambda g}(u) - g(u) \|^2 \leq 2\| g(u) \|^2 - 2(Y_{\lambda g}(u), g(u)) \to 0, \]
as \( \lambda \) tends to zero, which concludes the proof of the third assertion. \( \square \)

Lemma 2.2 Let \( \{ u_\lambda \} \) be a family of elements in \( H \), such that \( \lim_{\lambda \to 0} u_\lambda = u \in H \).

(a) Then \( \lim_{\lambda \to 0} R_{\lambda g}(u_\lambda) = u \).

(b) If \( u \in D(g) \) and \( \lim_{\lambda \to 0} (u - u_\lambda) / \lambda \) exists, then \( \lim_{\lambda \to 0} Y_{\lambda g}(u_\lambda) = g(u) \).

Proof. The first assertion follows by the inequality
\[ \| R_{\lambda g}(u_\lambda) - u \| = \| R_{\lambda g}(u_\lambda) - R_{\lambda g}(u) + R_{\lambda g}(u) - u \| \leq \| u_\lambda - u \| + \| R_{\lambda g}(u) - u \|. \]
To prove the second assertion, let \( z = \lim_{\lambda \to 0} (u - u_\lambda) / \lambda \), \( v_\lambda = u_\lambda + \lambda z \), and introduce the operator \( r \) given by \( r(u) = g(u) - z \). Hence, \( R_{\lambda r}(u_\lambda) = R_{\lambda r}(v_\lambda) \) and
\[ Y_{\lambda g}(u_\lambda) = \frac{1}{\lambda} (R_{\lambda r}(v_\lambda) - R_{\lambda r}(u)) + \frac{1}{2} (R_{\lambda r}(u) - u) + \frac{1}{2} (u - u_\lambda) = \frac{1}{\lambda} (R_{\lambda r}(v_\lambda) - R_{\lambda r}(u)) + Y_{\lambda r}(u) + \frac{1}{2} (u - u_\lambda). \]
This together with Lemma 2.1 yields the limit
\[ \| Y_{\lambda g}(u_\lambda) - g(u) \| \leq \| \frac{1}{\lambda} (v_\lambda - u) \| + \| Y_{\lambda r}(u) - r(u) \| + \| \frac{1}{2} (u - u_\lambda) - z \| \to 0, \]
as \( \lambda \) tends to zero. \( \square \)

The second assertion of Lemma 2.2 is originally due to (Lions & Mercier, 1979, Lemma 3). A last cornerstone of our analysis is to relate the full operator \( f \) with the sum \( f_a + f_b \). This can be achieved by the following assumption.
Assumption 2.2 The range of \( I - \lambda (fa + fb) \) is dense in \( H \) for all \( \lambda > 0 \).

Lemma 2.3 If Assumptions 2.1 and 2.2 are valid, then \( fa + fb = f \), i.e., \( \text{graph}(fa + fb) = \text{graph}(f) \).

Proof. Since every maximal dissipative operator is closed (Barbu, 1976, Proposition II.3.4), we have \( fa + fb \subseteq f \). Assumption 2.2 implies that there exists a sequence \( \{v_h\} \) in the range of \( I - \lambda (fa + fb) \) for every \( u \in D(f) \), such that \( \|u - v_h\| = o(\lambda) \). Let \( u_h = R_{fa}(v_h) \in D(fa) \cap D(fb) \), then Lemma 2.2 gives that \( \lim_{\lambda \to 0} u_h = u \) and \( \lim_{\lambda \to 0} fb(u_h) = \lim_{\lambda \to 0} R_{fa}(v_h) = f(u) \). Hence, \( f \subseteq fa + fb \), which concludes the proof. □

Note that the construction of \( u_h \) in the above proof also yields that the set \( D(fa) \cap D(fb) \) is dense in \( H \), whenever Assumptions 2.1 and 2.2 are valid.

3. Convergence of splitting schemes

Within this framework it is possible to address the question of convergence by employing the following result (Brézis & Pazy, 1972, Corollary 4.3).

Proposition 3.1 Let \( S(h) : H \to H \), with \( h > 0 \), be a family of nonexpansive operators. If

\[
\lim_{h \to 0} S(h) - I = f(u), \quad \text{for all } u \in D(fa) \cap D(fb),
\]

and Assumptions 2.1 and 2.2 hold, then

\[
\lim_{n \to \infty} S\left(\frac{1}{n}\right)^n (u_0) = e^f(u_0),
\]

for every \( u_0 \in H \), and the limit is uniform in \( t \) on bounded intervals.

With this result at hand one can prove convergence for a wide range of splitting schemes. As a first example consider the Lie splitting, where a single time step is given by

\[
S_t(h) = R_{fa} R_{fb}.
\]

Theorem 3.2 If Assumptions 2.1 and 2.2 are valid and \( u_0 \in H \), then

\[
\lim_{n \to \infty} S_t\left(\frac{1}{n}\right)^n (u_0) = e^f(u_0),
\]

and the limit is uniform in \( t \) on bounded intervals.

Proof. The operator \( S_t(h) \) is nonexpansive, by Lemma 2.1, and in virtue of Proposition 3.1 it remains to prove the consistency (3.1). To this end, let \( u_0 = R_{fb}(u) \) for an element \( u \in D(fb) \). Hence, \( \lim_{h \to 0} (u - u_h) / h = - \lim_{h \to 0} Y_{fb}(u) = - fb(u) \), and (3.1) follows by the equality

\[
\frac{1}{h} (S_t(h) - I)(u) = \frac{1}{h} (R_{fa} - I)(u_h) + \frac{1}{h} (u_h - u) = Y_{fa}(u) + Y_{fb}(u),
\]

together with Lemmas 2.1 and 2.2. □

It is also straightforward to prove that the “sum” splitting introduced in Coron (1982), with

\[
S_h = \frac{1}{h} (R_{fa} + R_{fb}),
\]

is convergent for all initial values \( u_0 \) in \( H \). To illustrate that the framework is applicable to (potentially) higher order schemes, consider the Peaceman–Rachford splitting, where

\[
S_{pr}(h) = R_{fa} (I + h fb) R_{fb} (I + h fa).
\]
Inspired by (Lions & Mercier, 1979, Theorem 2), we will prove convergence by introducing an intermediate (low order) scheme

\[ S_{pre}(h) = (I + hf_a) R_{bf_a} (I + hf_b) R_{bf_b} = (2R_{bf_a} - I)(2R_{bf_b} - I), \]

which relates to \( S_{pre} \) in the following fashion,

\[ S_{pre}(h)^n = R_{bf_b} S_{pre}(h)^n (I - hf_b). \] (3.2)

Note that, in contrast to our current setting, the convergence analysis of Lions & Mercier (1979) is restricted to the case when the operator \( f_a + f_b \) itself is maximal dissipative.

**Theorem 3.3** If Assumptions 2.1 and 2.2 are valid and \( u_0 \in D(f_b) \), then

\[ \lim_{n \to \infty} S_{pre} \left( \frac{t}{n} \right)^n (u_0) = e^{2f}(u_0), \]

and the limit is uniform in \( t \) on bounded intervals.

**Proof.** The operator \( S_{pre} \) is nonexpansive, as

\[
\| (I + g)(u) - (I + g)(v) \|^2 = \| u - v \|^2 + 2(g(u) - g(v), u - v) + \| g(u) - g(v) \|^2 \\
\leq \| u - v \|^2 - 2(g(u) - g(v), u - v) + \| g(u) - g(v) \|^2 \\
= \| (I - g)(u) - (I - g)(v) \|^2,
\]

for every maximal dissipative \( g \). Furthermore, with \( u \in D(f_a) \cap D(f_b) \) and \( u_h = (2R_{bf_a} - I)(u) \), we obtain the consistency (3.1) of \( S_{pre} \) via the equality

\[
\frac{1}{h}(S_{pre}(h) - I)(u) = \frac{1}{h}(2R_{bf_a}(u_h) - u_h - u) = \frac{1}{h}(R_{bf_a} - I)(u_h) + h(u_h - u) = 2Y_{bf_a}(u_h) + 2Y_{bf_b}(u).
\]

Hence, Proposition 3.1 implies that

\[
\lim_{n \to \infty} S_{pre} \left( \frac{t}{n} \right)^n (u_0) = e^{2f}(u_0),
\]

for every \( u_0 \in H \). We next restrict the initial value \( u_0 \in D(f_b) \) and let \( u_h = S_{pre}(h)^n (I - hf_b)(u_0) \). As \( S_{pre}(h) \) is nonexpansive, one has

\[
\| u_h - S_{pre}(h)^n(u_0) \| \leq h \| f_b(u_0) \|,
\]

for all \( h > 0 \). This yields that \( \lim_{n \to \infty} u_{t/n} = e^{2f}(u_0) \), and Lemma 2.2 gives us the desired convergence,

\[
\lim_{n \to \infty} S_{pre} \left( \frac{t}{n} \right)^n (u_0) = \lim_{n \to \infty} R_{bf_b}(u_{t/n}) = e^{2f}(u_0).
\]

As a last remark, we note that the convergence of the Lie splitting can be extended to a Banach space framework, Barbu (1976); Brézis & Pazy (1972), whereas, there is little hope to generalize the Peaceman–Rachford results to arbitrary Banach spaces, as the nonexpansivity of the time stepping operator \( S_{pre} \) is in general lost even for linear problems, see Schatzman (1999).
4. Dimension splitting for quasilinear equations

The aim is now to validate that the abstract framework developed in Section 2 is applicable for quasilinear elliptic problems, and we start by treating the following classical example.

Consider the parabolic equation (2.1) equipped with Dirichlet boundary conditions and governed by the quasilinear vector field $f$, where

$$\begin{align*}
f(u) = (f_a + f_b)(u) &= \sum_{i=1}^{s} D_i a_i(x, D_i u) + c_i D_i u + g_i(x) \\
&\quad + \sum_{j=r+1}^{d} D_j a_j(x, D_j u) + c_j D_j u + g_j(x)
\end{align*}$$

for sufficiently regular functions $u$. Here, $x = (x_1, \ldots, x_d) \in \Omega$, $D_i = \partial/\partial x_i$, $c_i \in \mathbb{R}$, $g_i \in L^2(\Omega)$, and $\Omega$ is a bounded domain in $\mathbb{R}^d$ with a locally Lipschitz continuous boundary $\partial \Omega$. We furthermore assume that the functions $a_i : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ are continuously differentiable and satisfy the bounds

$$0 < m \leq D_2 a_i(\xi, z) \leq M \quad \text{for all } (\xi, z) \in \mathbb{R}^d \times \mathbb{R}. \quad (4.1)$$

To cast this into the framework of Section 2, let $H = L^2(\Omega)$ and introduce the operator $f_s$ defined as

$$f_s(u) = \sum_{i=1}^{s} D_i a_i(x, D_i u) + c_i D_i u + g_i(x), \quad (4.2)$$

where $s \leq d$. Since $f$, $f_a$ and $f_b$ are all of the form (4.2), after a renumbering of the coefficients, it is sufficient to consider $f_s$ in the analysis. To do so, define the real Hilbert space $V_s$ which is the completion of $C_0^\infty(\Omega)$ with respect to the norm induced by the inner product

$$(u, v)_{V_s} = \sum_{i=1}^{s} (D_i u, D_i v)_{L^2(\Omega)} + (u, v)_{L^2(\Omega)}. \quad (4.3)$$

By the standard line of reasoning (Friedman, 1982, p.104), it follows that the space $V_s$ consists of $L^2$ functions, for which the first $s$ partial distributional derivatives are again $L^2$ elements. One also obtains the following chain of imbeddings

$$V_s \hookrightarrow L^2(\Omega) \cong L^2(\Omega)' \hookrightarrow V_s', \quad (4.4)$$

where $V_s'$ denotes the dual space of $V_s$. Next, let $\langle \cdot , \cdot \rangle$ be the dual paring between $V_s'$ and $V_s$, and consider the weak, or variational, version of the operator $f_s$, namely $F_s : V_s \rightarrow V_s'$ which is given by

$$\langle F_s(u), v \rangle = \sum_{i=1}^{s} \langle a_i(\cdot, D_i u) + c_i u, D_i v \rangle_{L^2(\Omega)} + \langle g_i, v \rangle_{L^2(\Omega)} \quad \text{for all } v \in V_s.$$

**Lemma 4.1** The operator $F_s$ is monotone and demicontinuous, i.e., $u_n \rightharpoonup u$ implies that $F_s(u_n) \rightharpoonup (F_s(u), v)$ for all $v \in V_s$.

**Proof.** **Monotonicity:** Let $u, v \in V_s$ and $u_t = tD_t u + (1-t)D_tv$, then

$$\langle F_s(u) - F_s(v), u - v \rangle \geq \sum_{i=1}^{s} \left( \int_0^1 \int_{\Omega} D_i a_i(\cdot, u_t) D_i(u - v), D_i(u - v) \right)_{L^2(\Omega)}$$

$$\geq m \sum_{i=1}^{s} \|D_i(u - v)\|_{L^2(\Omega)}^2 \geq 0.$$
Note that the advection terms \( c_i D_i u \) drop in the above inequality due to the employed Dirichlet boundary conditions.

**Demicontinuity:** If \( u_n \to u \) in \( V \), then we have the limit

\[
|\langle F_i(u) - F_i(u_n), v \rangle| \leq \sum_{i=1}^{r} \left( M_\Omega \| D_i(u - u_n) \|_{L^2(\Omega)} + |c_i| \| u - u_n \|_{L^2(\Omega)} \right) \| D_i v \|_{L^1(\Omega)}
\]

\[
\leq C \| u - u_n \|_{V} \| v \|_{V} \to 0,
\]

for every \( v \in V \).

**Lemma 4.2** Let \( I \) be the identity on \( L^2(\Omega) \). The map \( I + \lambda F_i : V \to V' \) is onto for every \( \lambda > 0 \).

**Proof.** By Lemma 4.1, the operator \( I + \lambda F_i \) is demicontinuous and

\[
\langle (I + \lambda F_i)(u) - (I + \lambda F_i)(v), u - v \rangle \geq \min(1, m\lambda) \| u - v \|_{V}^2,
\]

for every \( u, v \in V \). Hence, \( I + \lambda F_i \) is strongly monotone and therefore also coercive. The surjectivity then follows by the Browder and Minty theorem; see for example (Zeidler, 1990, Theorem 26.A).

We can finally interpret \( f_i \) as an operator on \( L^2(\Omega) \), by defining

\[
D(f_i) = \{ u \in V : \exists \xi \in L^2(\Omega) \text{ with } (z, v)_{L^2(\Omega)} = -\langle F_i(u), v \rangle \forall v \in V \}
\]

and \( f_i(u) = z \) for all \( u \in D(f_i) \). Since the functions \( a_i \) are assumed to be continuous differentiable, it holds that \( \{ u \in C^2(\Omega) \cap C^0(\overline{\Omega}) : u = 0 \text{ on } \partial \Omega \} \subset D(f_i) \), and \( f_i \) is therefore densely defined on \( L^2(\Omega) \).

Assumption (2.1) now follows by the lemma below.

**Lemma 4.3** The operator \( f_i \) is maximal dissipative on \( L^2(\Omega) \).

**Proof.** Dissipativity: For every \( u, v \in D(f_i) \),

\[
\langle f_i(u) - f_i(v), u - v \rangle_{L^2(\Omega)} = -\langle F_i(u) - F_i(v), u - v \rangle \leq 0.
\]

**Maximality:** The operator \( I + \lambda F_i : V \to V' \) is onto, by Lemma 4.2, and \( L^2(\Omega)' \hookrightarrow V' \). Hence, for every \( z \in L^2(\Omega) \) and \( \lambda > 0 \) there exists an element \( u \in V \) such that

\[
-\langle F_i(u), v \rangle = \frac{1}{\lambda} (u - z, v)_{L^2(\Omega)},
\]

for all \( v \in V \). This implies that \( u \in D(f_i) \) and \( f_i(u) = \frac{1}{\lambda}(u - z) \), i.e., \( I - \lambda f_i \) is onto. □

To conclude that the derived convergence results of Section 3 are applicable for the current quasilinear vector field, we observe that Assumption 2.2 holds true whenever the space \( C_0^\infty(\Omega) \) is a subset of \( R(I - \lambda (f_a + f_b)) \), or in other words, the elliptic equation

\[
(I - \lambda (f_a + f_b))(u) = h \in C_0^\infty(\Omega)
\]

has a solution \( u \) for which the terms \( D_a u, D_b u \) are all elements in \( L^2(\Omega) \).

**Example 4.1** Let \( d = 2 \) and assume that \( \partial \Omega \) is sufficiently regular. If in addition \( a_i \in C^{1,\beta}(\Omega \times \mathbb{R}) \) and \( g_i \in C^\beta(\Omega) \) for some \( \beta \in (0, 1) \), then Assumption 2.2 is valid as

\[
R_{\lambda f} C_0^\infty(\Omega) \subseteq \{ u \in C^{1,\beta}(\Omega) \cap C^0(\overline{\Omega}) : u = 0 \text{ on } \partial \Omega \}.
\]

The proof follows by the standard elliptic theory, see for instance (Gilbarg & Trudinger, 1983, Theorem 12.5).
5. Dimension splitting for degenerate equations

The analysis presented in the previous sections is in fact fully nonlinear and the arguments can be generalized to larger classes of diffusion coefficients $a_i$. One possible extension is to replace (4.1) by

$$0 < m|z - \tilde{z}|^p \leq \left(a_i(\xi, z) - a_i(\xi, \tilde{z})\right)(z - \tilde{z}) \quad \text{and} \quad |a_i(\xi, z)| \leq M(1 + |z|^{p-1}), \quad \text{for all } (\xi, z, \tilde{z}) \in \overline{\Omega} \times \mathbb{R}^2,$$

(5.1)

where $p \in [2, \infty)$. Here, the diffusion coefficients may degenerate, i.e., $D_2a_i(x, z)$ can be zero at some instances. One of the standard examples is

$$a_i(x, z) = |z|^{p-2}z.$$

Since there is no far-reaching regularity analysis for the present family of nonlinear parabolic equations, one has no means of giving a general approach to validate Assumption 2.2. We will therefore settle instances. One of the standard examples is

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$$0 < m|z - \tilde{z}|^p \leq \left(a_i(\xi, z) - a_i(\xi, \tilde{z})\right)(z - \tilde{z}) \quad \text{and} \quad |a_i(\xi, z)| \leq M(1 + |z|^{p-1}), \quad \text{for all } (\xi, z, \tilde{z}) \in \overline{\Omega} \times \mathbb{R}^2,$$

(5.1)

where $p \in [2, \infty)$. Here, the diffusion coefficients may degenerate, i.e., $D_2a_i(x, z)$ can be zero at some instances. One of the standard examples is

$$a_i(x, z) = |z|^{p-2}z.$$

Since there is no far-reaching regularity analysis for the present family of nonlinear parabolic equations, one has no means of giving a general approach to validate Assumption 2.2. We will therefore settle instances. One of the standard examples is

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$$a_i(x, z) = |z|^{p-2}z.$$
is also continuous, see (Zeidler, 1990, Proposition 26.7). Hence, the operator $F_s$ is demicontinuous, as

$$|\langle F_s(u) - F_s(u_n), v \rangle| \leq \sum_{i=1}^{s} \left( |a_i(\cdot, D_i u) - a_i(\cdot, D_i u_n)||u||_{L^p(\Omega)} + |c_i||u - u_n||_{L^p(\Omega)} \right) \|D_i v\|_{L^p(\Omega)}.$$  

From these results, it follows that the operator $I + \lambda F_s$ is also monotone and demicontinuous. As the space $V_s$ is reflexive, the desired surjectivity again follows by the Browder and Minty theorem (Zeidler, 1990, Theorem 26.A), whenever $I + \lambda F_s$ is coercive. To prove the latter, consider the lower bound

$$\langle (I + \lambda F_s)(u), u \rangle = \|u\|_{L^2(\Omega)}^2 + \lambda \sum_{i=1}^{s} \int_{\Omega} (a_i(x, D_i u) - a_i(x, 0)) D_i u + (g_i - D_i a_i(x, 0)) u \, dx$$  

$$\geq \min(1, m\lambda) \left( \sum_{i=1}^{s} \|D_i u\|_{L^p(\Omega)}^p + \|u\|_{L^2(\Omega)}^2 \right) - \lambda \sum_{i=1}^{s} \left( \|g_i\|_{L^2(\Omega)} + \|D_i a_i(\cdot, 0)\|_{L^2(\Omega)} \right) \|u\|_{L^2(\Omega)},$$  

which yields the limit

$$\frac{\langle (I + \lambda F_s)(u), u \rangle}{\|u\|_{V_s}} \geq \min(1, m\lambda) \sum_{i=1}^{s} \|D_i u\|_{L^p(\Omega)}^p + \|u\|_{L^2(\Omega)}^2 - \lambda \sum_{i=1}^{s} \left( \|g_i\|_{L^2(\Omega)} + \|D_i a_i(\cdot, 0)\|_{L^2(\Omega)} \right) \to \infty,$$

as $\|u\|_{V_s}$ tends to infinity, i.e., $I + \lambda F_s$ is coercive for every $\lambda > 0$. 

In conclusion, we can once again interpret $f_s$ as an operator on $L^2(\Omega)$ via $F_s$. Assumption 2.1 then follows by repeating the proof of Lemma 4.3, for which Lemma 5.2 is needed, and noting that $\{u \in C^2(\Omega) \cap C^0(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\} \subset D(f_s).$

6. Numerical experiments

To illustrate the convergence results derived in Theorems 3.2 and 3.3, we consider the parabolic equation over $\Omega = (0, 1)^2$, equipped with Dirichlet boundary conditions, and governed by the quasilinear vector field

$$f(u) = (f_a + f_b)(u) = (D_1 a(D_1 u) + cD_2 u + \frac{1}{2}g_1(x)) + (D_2 a(D_2 u) + cD_2 u + \frac{1}{2}g_2(x)).$$

The numerical experiments are conducted in two cases. First for the reaction-diffusion-advection field with

$$a(z) = (1 + e^{-z^2})z, \quad c = -3 \quad \text{and} \quad g(x) = 10 \chi_{(1/2, 1/2)}(x), \quad (6.1)$$

where $\chi_{\Omega_0}$ is the characteristic function of $\Omega_0$, and secondly for the degenerate vector field given by

$$a(z) = |z|^2z \quad \text{and} \quad c = g(x) = 0. \quad (6.2)$$

Note that the diffusion coefficients in (6.1) and (6.2), satisfy the bounds (4.1) and (5.1), respectively.

The vector fields are discretized by the standard central difference scheme, and the related discrete operators $f^h_a$ and $f^h_b$ are then defined as

$$\left( f^h_a u \right)_{i,j} = \frac{1}{h} \left( a \left( \frac{1}{h} (u_{i+1,j} - u_{i,j}) \right) - a \left( \frac{1}{h} (u_{i,j} - u_{i-1,j}) \right) \right) + c \frac{1}{h^2} (u_{i+1,j} - u_{i,j-1}) + \frac{1}{2} g_{i,j} \quad \text{and} \quad \left( f^h_b u \right)_{i,j} = \frac{1}{h} \left( a \left( \frac{1}{h} (u_{i,j+1} - u_{i,j}) \right) - a \left( \frac{1}{h} (u_{i,j} - u_{i,j-1}) \right) \right) + c \frac{1}{h^2} (u_{i,j+1} - u_{i,j-1}) + \frac{1}{2} g_{i,j},$$

where $a$ and $g$ are as defined in (6.1) and (6.2), respectively.
where \( i, j = 1, \ldots, m, k = 1/(m+1) \) and \( g_{t,j} = g(i,k,jk) \). Due to the boundary conditions, we have
\[
  u_{0,j} = u_{m+1,j} = u_{i,0} = u_{i,m+1} = 0.
\]

The underlying algebraic equations, resulting from the terms \( R_k f_k \) and \( R_k g_k \), are approximated by Newton’s method. Moreover, the errors introduced by the splitting schemes are estimated, at time \( t \), by comparing the numerical result with the one obtained for \( n_{ref} = 2^{13} \) time steps. In the tests of the Lie splitting we choose an initial value \( u_0 \) in \( H = L^2(\Omega) \), which is not differentiable, and for the Paceman–Rachford tests we use \( u_0 \in \{ u \in C^2(\Omega) \cap C^0(\partial \Omega) : u = 0 \text{ on } \partial \Omega \} \subset D(f_k) \). The initial values, their evolutions by the degenerate vector field (6.2) with the parameter choice \([t, m, n] = [0.1, 60, 50]\), and the related absolute pointwise errors are all presented in Figure 1.

To investigate the convergence of the splitting schemes, we approximate the error in the discrete \( L^2 \) norm for different time steps \( n \). The obtained numerical errors and orders, for \([t, m] = [0.1, 100]\), are given in Table 1. The results clearly illustrate that the convergence derived in Theorems 3.2 and 3.3 is obtained for the two splittings in both the nondegenerate and the degenerate case. Rather surprisingly, the simulations of the degenerate problem even display the classical convergence orders found when considering linear evolutions. We are, however, currently unaware of any nonlinear analysis that allows the derivation of such explicit convergence orders.

| Table 1. The numerical errors and orders observed in the simulations. |
|-------------------------|-------------------------|
| **Evolution (6.1)**     | **Evolution (6.2)**     |
| \( n \)                | \( n \)                | \( \text{error} \) | \( \text{order} \) | \( \text{error} \) | \( \text{order} \) | \( \text{error} \) | \( \text{order} \) |
| 8                      | 7.36 e-3               | 1.97 e-4           |                          |                          |                          |                          |                          |
| 16                     | 4.09 e-3               | 0.85               | 4.36 e-5                | 2.17                      |                          |                          |                          |
| 32                     | 2.18 e-3               | 0.91               | 1.08 e-5                | 2.02                      |                          |                          |                          |
| 64                     | 1.13 e-3               | 0.95               | 2.66 e-6                | 2.01                      |                          |                          |                          |
| 128                    | 5.70 e-4               | 0.98               | 6.62 e-7                | 2.01                      |                          |                          |                          |
| 256                    | 2.84 e-4               | 1.01               | 1.65 e-7                | 2.00                      |                          |                          |                          |
| 512                    | 1.38 e-4               | 1.04               | 4.11 e-8                | 2.00                      |                          |                          |                          |
| 1024                   | 6.45 e-5               | 1.10               | 1.02 e-8                | 2.02                      |                          |                          |                          |
| **Lie**                |                        |                    |                          |                          |                          |                          |                          |
| **Peaceman–Rachford**  |                        |                    |                          |                          |                          |                          |                          |

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**REFERENCES**

The figure displays a few of the results obtained from our simulations. The graphs in the left column are from the experiments with the Lie splitting (top to bottom: initial value \( u_0 \), the evolution of \( u_0 \) at time \( t = 0.1 \) with respect to the degenerate field (6.2), and the related absolute pointwise error for \([m,n] = [60,50]\)). Graphs to the right display the corresponding results for the Peaceman–Rachford splitting.


