7.0: Minimax approximations

In this section we study the problem

$$\min_{p \in \mathcal{A}} \|f - p\|_{\infty} = \min_{p \in \mathcal{A}} \max_{x \in [a,b]} |f(x) - p(x)|$$

where $f \in C[a,b]$ and $\mathcal{A}$ is a linear subspace of $C[a,b]$.

Let $p^*$ be a trial solution (e.g. a guess) of this problem. To find a solution one might seek for a $p \in \mathcal{A}$ such that

$$\|f - (p^* + p)\|_{\infty} < \|f - p^*\|_{\infty}$$

We have to study where the error function $e = f - p$ takes a maximum value.
7.1: Example
7.2: The extremal set $\mathcal{Z}_M$

Let $p^* \in \mathcal{A}$ be a trial function and $e^*(x) = f(x) - p^*(x)$ its error function.

Let

$$\mathcal{Z}_M = \{\xi \in [a, b] : |e^*(\xi)| = \max_{x \in [a, b]} |e^*(x)| = \|e^*\|_{\infty}\}.$$

Then, if $p^* + p$ is the best approximation

$$|e^*(x) - p(x)| < |e^*(x)| \quad \forall x \in \mathcal{Z}_M.$$

Consequently, $e^*(x)$ and $p(x)$ have the same sign for all $x \in \mathcal{Z}_M$. 

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7.3: A sign criterion

We conclude

If there is no $p \in A$ such that $(f(x) - p^*(x))p(x) > 0 \ \forall x \in Z_M$, then $p^*$ is best approximation.

The next theorem states even the “$\Leftarrow$”-direction so that the sign criterion

There is no $p \in A$ such that $(f(x) - p^*(x))p(x) > 0 \ \forall x \in Z_M \quad (*)$

becomes a necessary and sufficient condition for a best minimax approximation.

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7.4: Best minimax approximation theorem

Theorem. [cf. Th. 7.1]
Let \( A \subset C[a, b] \) and \( f \in C[a, b] \). Let \( Z \) be an arbitrary closed subset of \([a, b]\) and \( p^* \in A \) be an arbitrary trial function.

Let furthermore \( Z_M = \{ \xi \in Z : |e^*(\xi)| = \|e^*\|_\infty \} \).

Then
\[
\max_{x \in Z} |f(x) - p^*(x)| \leq \max_{x \in Z} |f(x) - p(x)| \quad \forall p \in A
\]

if and only if

there is no \( p \in A \) such that \((f(x) - p^*(x))p(x) > 0 \) \( \forall x \in Z_M \) \( (*) \)
7.5: The polynomial case

Let $\mathcal{A} = \mathcal{P}^n$.

Case 1:
$(f(x) - p^*(x))$ changes in $\mathbb{Z}_M$ more than $n$-times its sign. No polynomial in $\mathcal{P}^n$ can change its sign more then $n$ times. Thus, there must exist at least one $x$ where $p$ and $f(x) - p^*(x)$ have different signs. Consequently condition $(\ast)$ is fulfilled and $p^*$ is best approximation.

Case 2:
$(f(x) - p^*(x))$ changes in $\mathbb{Z}_M$ at most $n$-times its sign. Then we can construct a nonzero polynomial $p \in \mathcal{P}^n$, which has its sign changes at exactly the same points. Condition $(\ast)$ is not fullfilled and $p^*$ is not a best approximation.
7.6: Haar condition

This observation holds even for a slightly more general class of functions than polynomials – functions which fullfill the Haar-condition:

**Definition. [p. 77]** Let $\mathcal{A} \subset C[a, b]$ be an $n+1$ dimensional linear subspace with elements $p$ which have the property:
If $p$ has more than $n$ zeros in $[a, b]$ than it is identical zero.
Then $\mathcal{A}$ is called a Haar space or a space which fullfills the Haar condition.
A basis in a Haar space $\mathcal{A}$ is called a Chebychev set.
7.7: Haar condition – equivalent properties

Theorem. The Haar condition is equivalent to one of the following conditions

- If $\{\xi_i, i = 0, \ldots, n\} \subset [a, b]$ are distinct and $\{\phi_i, i = 0, \ldots, n\}$ is a basis of $\mathcal{A}$, then the matrix $V = (\phi_i(\xi_j))_{i,j}$ is non singular.

- If $0 \neq p \in \mathcal{A}$ has $j$ zeros of which $k$ zeros $\xi_i$ are in $(a, b)$ and $p$ does not change sign at $\xi_i$, then $j + k \leq n$.

Furthermore, if $\mathcal{A}$ is a Haar space then the following statement holds: If $\{\xi_i, i = 0, \ldots, k\} \subset [a, b]$ are distinct and $k \leq n$, then there exists a $p \in \mathcal{A}$, which changes its sign at $\xi$ and has no other zeros. In particular, there is a $p \in \mathcal{A}$ which has no zeros at all in $[a, b]$.

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7.7: Characterization Theorem

Theorem. [cf. Th. 7.2, Characterization Theorem]
Let $A \subset C[a, b]$ be an $n + 1$-dimensional Haar space and let $f \in C[a, b]$. Then

$p^* \in A$ is the best minimax approximation of $f \in A \Leftrightarrow$

there exists an ordered set of distinct points \{\(\xi_i^* : i = 0, \ldots, n + 1\)\} $\subset [a, b]$, with

\[
|f(\xi_i^*) - p^*(\xi_i^*)| = \|f - p^*\|_\infty \quad i = 0, \ldots, n + 1 \quad \text{and}
\]

\[
f(\xi_{i+1}^*) - p^*(\xi_{i+1}^*) = -(f(\xi_i^*) - p^*(\xi_i^*)) \quad i = 0, \ldots, n
\]
7.8: Minimal property of Chebychev polynomials

**Theorem. [cf. Th. 7.3]**
Let \( P_1^n \) be the set of all polynomials of degree \( n \) with leading coefficient \( c_n = 1 \) and let \( \frac{1}{2}^{n-1} T_n \in P_1^n \) be the scaled Chebychev polynomial of degree \( n \). Then,
\[
\| \frac{1}{2}^{n-1} T_n \|_\infty = \min_{p \in P_1^n} \| p \|_\infty
\]

For the proof, we rewrite \( p \in P_1^n \) as
\[
\| p \|_\infty = \| x^n_f - \sum_{i=0}^{n-1} c_i x_i \|_\infty
\]
and use the properties of the Chebychev polynomial to apply the Characterization theorem.

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7.9: Minimality on a reference

We call a set of $n + 2$ points $a \leq \xi_0 < \xi_1 < \cdots < \xi_{n+1} \leq b$ a reference.

**Theorem. [cf. Th. 7.4]**

Let $\{\xi_i, i = 0, \ldots, n + 1\}$ be a reference, $f \in C[a, b]$ and $A$ an $(n + 1)$-dimensional Haar space. Then,

$$\max_{i=0,\ldots,n+1} |f(\xi_i) - p^* (\xi_i)| \leq \max_{i=0,\ldots,n+1} |f(\xi_i) - p(\xi_i)| \quad \forall p \in A$$

$$\iff$$

$$f(\xi_{i+1}) - p^*(\xi_{i+1}) = -(f(\xi_i) - p^*(\xi_i)) \quad i = 0, \ldots, n$$
7.10: Example

We consider \( f(x) = \sin(x) \) in \([0, 2.5]\) and construct a polynomial \( p(x) = a_2 x^2 + a_1 x + a_0 \) of degree 2 which is minimal on the reference \( Z = 0.5, 1, 1.5, 2.0 \).

For this end we define \( h := f(\xi_0) - p(\xi_0) = f(0.5) - p(0.5) \) and set up a linear equation system for \( a_0, a_1, a_2 \) and \( h \) in such a way that the condition from Theorem 7.4 is fullfilled:

\[
\begin{align*}
\sin(0.5) - a_2 0.5^2 - a_1 0.5 - a_0 &= h \\
\sin(1.0) - a_2 - a_1 - a_0 &= -h \\
\sin(1.5) - a_2 1.5^2 - a_1 1.5 - a_0 &= h \\
\sin(2.0) - a_2 2.0^2 - a_1 2.0 - a_0 &= -h
\end{align*}
\]
7.11: Example

Points where the sign condition is applied (see Th. 7.4)

levelled reference error $h$: $h = 4.8 \times 10^{-3}$

$e(x) = f(x) - p(x)$
7.12: Uniqueness of a best minimax approximation

A point $\xi \in (a, b)$ with $f(\xi) = 0$ is called a double zero if $f$ does not change its sign at $\xi$. (Note, this definition calls zeros which classically are considered to have a higher but even multiplicity as "double" zeros.)

In the sequel we assume the existence of two best solutions $p^*$ and $q^*$ and set $r = q^* - p^*$. A uniqueness proof has to show that $r \equiv 0$.

**Theorem.** [cf. Th. 7.5]

Let $r \in C[a, b]$ and let $\mathcal{Z} = \{\xi_i, i = 0, \ldots, n + 1\}$ be a reference, with

\[ (-1)^i r(\xi_i) \geq 0 \quad \forall \xi_i \in \mathcal{Z}. \]

Then $r$ has at least $n + 1$ zeros in $[a, b]$ where double zeros are counted twice.
7.11: Uniqueness of a best minimax approximation (2)

**Theorem. [cf. Th. 7.6]**

Let $\mathcal{A} \subset C[a, b]$ be a Haar space. Then there exists for any $f \in C[a, b]$ a unique best minimax approximation in $\mathcal{A}$.

Note, that the Haar condition is a necessary condition. There are examples for non Haar spaces which give multiple best approximations.

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7.13: A lower bound

Let a function $p^*$ share the sign alternation property with the best approximation. Can we deduce from that a statement about the quality of the best approximation?

**Theorem. [cf. Th. 7.7]**

Let $\mathcal{A}$ be an $(n + 1)$ dimensional Haar space which fullfills the condition of the Characterization Theorem. For $p^* \in \mathcal{A}$ with the sign property

$$\text{sign} \left( f(\xi_{i+1}) - p^*(\xi_{i+1}) \right) = -\text{sign} \left( f(\xi_i) - p^*(\xi_i) \right) \quad \forall \xi_i \in \mathcal{Z},$$

where $\mathcal{Z} = \{\xi_i, i = 0, \ldots, n + 1\}$ is a reference, we get:

$$\min_{i=0:n+1} |f(\xi_i) - p^*(\xi_i)| \leq \min_{p \in \mathcal{A}} \max_{i=0:n+1} |f(\xi_i) - p(\xi_i)| \leq \min_{p \in \mathcal{A}} \|f - p\|_{\infty} \leq \|f - p^*\|_{\infty}$$