7.0: Minimax approximations

In this section we study the problem

\[
\min_{p \in \mathcal{A}} \|f - p\|_{\infty} = \min_{p \in \mathcal{A}} \max_{x \in [a,b]} |f(x) - p(x)|
\]

where \( f \in C[a,b] \) and \( \mathcal{A} \) is a linear subspace of \( C[a,b] \).

Let \( p^* \) be a trial solution (e.g. a guess) of this problem. To find a solution one might seek for a \( p \in \mathcal{A} \) such that

\[
\|f - (p^* + p)\|_{\infty} < \|f - p^*\|_{\infty}
\]

We have to study where the error function \( e = f - p \) takes a maximum value.

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7.1: Example
Let \( p^* \in A \) be a trial function and \( e^*(x) = f(x) - p^*(x) \) its error function.

Let

\[
\mathcal{Z}_M = \{ \xi \in [a, b] : |e^*(\xi)| = \max_{x \in [a, b]} |e^*(x)| = \|e^*\|_\infty \}.
\]

Then, if \( p^* + p \) is the best approximation

\[
|e^*(x) - p(x)| < |e^*(x)| \quad \forall x \in \mathcal{Z}_M.
\]

Consequently, \( e^*(x) \) and \( p(x) \) have the same sign for all \( x \in \mathcal{Z}_M \).
7.3: A sign criterion

We conclude

If there is no \( p \in \mathcal{A} \) such that \((f(x) - p^*(x))p(x) > 0 \ \forall x \in \mathbb{Z}_M\), then \( p^* \) is best approximation.

The next theorem states even the “\( \Leftarrow \)”-direction so that the sign criterion

\[
\text{There is no } p \in \mathcal{A} \text{ such that } (f(x) - p^*(x))p(x) > 0 \ \forall x \in \mathbb{Z}_M \quad (\ast)
\]

becomes a necessary and sufficient condition for a best minimax approximation.

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7.4: Best minimax approximation theorem

Theorem. [cf. Th. 7.1]
Let $\mathcal{A} \subset C[a, b]$ and $f \in C[a, b]$. Let $\mathcal{Z}$ be an arbitrary closed subset of $[a, b]$ and $p^* \in \mathcal{A}$ be an arbitrary trial function.
Let furthermore $\mathcal{Z}_M = \{\xi \in \mathcal{Z} : |e^*(\xi)| = \|e^*\|_\infty\}$.
Then
$$\max_{x \in \mathcal{Z}} |f(x) - p^*(x)| \leq \max_{x \in \mathcal{Z}} |f(x) - p(x)| \quad \forall p \in \mathcal{A}$$

if and only if

there is no $p \in \mathcal{A}$ such that $(f(x) - p^*(x))p(x) > 0 \quad \forall x \in \mathcal{Z}_M \quad (*)$
7.5: The polynomial case

Let $\mathcal{A} = \mathcal{P}^n$.

Case 1:
$(f(x) - p^*(x))$ changes in $\mathcal{Z}_M$ more than $n$-times its sign. No polynomial in $\mathcal{P}^n$ can change its sign more then $n$ times. Thus, there must exist at least one $x$ where $p$ and $f(x) - p^*(x)$ have different signs. Consequently condition ($\ast$) is fulfilled and $p^*$ is best approximation.

Case 2:
$(f(x) - p^*(x))$ changes in $\mathcal{Z}_M$ at most $n$-times its sign. Then we can construct a nonzero polynomial $p \in \mathcal{P}^n$, which has its sign changes at exactly the same points. Condition ($\ast$) is not fullfilled and $p^*$ is not a best approximation.
7.6: Haar condition

This observation holds even for a slightly more general class of functions than polynomials – functions which fulfill the Haar-condition:

**Definition. [p. 77]** Let \( A \subset C[a, b] \) be a linear subspace with elements \( p \) which have the property:

*If \( p \) has more than \( n \) zeros in \([a, b]\) than it is identical zero. Then \( A \) is called a Haar space or a space which fulfills the Haar condition. A basis in a Haar space \( A \) is called a Chebychev set.*
7.7: Haar condition – equivalent properties

**Theorem.** The Haar condition is equivalent to one of the following conditions

- If \( \{ \xi_i, i = 0, \ldots, n \} \subset [a, b] \) are distinct and \( \{ \phi_i, i = 0, \ldots, n \} \) is a basis of \( \mathcal{A} \), then the matrix \( V = (\phi_i(\xi_j))_{i,j} \) is non singular.

- If \( 0 \neq p \in \mathcal{P}^n \) has \( j \) zeros of which \( k \) zeros \( \xi_i \) are in \( (a, b) \) and \( p \) does not change sign at \( \xi_i \), then \( j + k \leq n \).

Furthermore, if \( \mathcal{A} \) is a Haar space then the following statement holds:
If \( \{ \xi_i, i = 0, \ldots, k \} \subset [a, b] \) are distinct and \( k \leq n \), then there exists a \( p \in \mathcal{A} \), which changes its sign at \( \xi \) and has no other zeros. In particular, there is a \( p \in \mathcal{A} \) which has no zeros at all in \( [a, b] \).
7.7: Characterization Theorem

Theorem. [cf. Th. 7.2, Characterization Theorem]
Let \( A \subset C[a, b] \) be an \( n + 1 \)-dimensional Haar space and let \( f \in C[a, b] \).
Then

\[ p^* \in A \text{ is the best minimax approximation of } f \in A \iff \]

there exists an ordered set of distinct points \( \{\xi_i^* : i = 0, \ldots, n + 1\} \subset [a, b] \), with

\[ |f(\xi_i^*) - p^*(\xi_i^*)| = \|f - p^*\|_\infty \quad i = 0, \ldots, n + 1 \]

and

\[ f(\xi_{i+1}^*) - p^*(\xi_{i+1}^*) = - (f(\xi_i^*) - p^*(\xi_i^*)) \quad i = 0, \ldots, n \]
7.8: Minimal property of Chebychev polynomials

Theorem. [cf. Th. 7.3]
Let $P^n_1$ be the set of all polynomials of degree $n$ with leading coefficient $c_n = 1$ and let $\frac{1}{2}^{n-1} T_n \in P^n_1$ be the scaled Chebychev polynomial of degree $n$. Then,

$$\|\frac{1}{2}^{n-1} T_n\|_\infty = \min_{p \in P^n_1} \|p\|_\infty$$

For the proof, we rewrite $p \in P^n_1$ as

$$\|p\|_\infty = \|x^n_f - \sum_{i=0}^{n-1} c_i x_i\|_\infty$$

and use the properties of the Chebychev polynomial to apply the Characterization theorem.

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7.9: Minimality on a reference

We call a set of \( n + 2 \) points \( a \leq \xi_0 < \xi_1 < \cdots < \xi_{n+1} \leq b \) a reference.

**Theorem.** [cf. Th. 7.4]

Let \( \{\xi_i, i = 0, \ldots, n+1\} \) be a reference, \( f \in C[a, b] \) and \( \mathcal{A} \) an \((n + 1)\)-dimensional Haar space.

Then,

\[
\max_{i=0,\ldots,n+1} |f(\xi_i) - p^*(\xi_i)| \leq \max_{i=0,\ldots,n+1} |f(\xi_i) - p(\xi_i)| \quad \forall p \in \mathcal{A}
\]

\[
\iff
\]

\[
f(\xi_{i+1}) - p^*(\xi_{i+1}) = -(f(\xi_i) - p^*(\xi_i)) \quad i = 0, \ldots, n
\]
7.10: Example

We consider \( f(x) = \sin(x) \) in \([0, 2.5]\) and construct a polynomial \( p(x) = a_2 x^2 + a_1 x + a_0 \) of degree 2 which is minimal on the reference \( Z = 0.5, 1, 1.5, 2.0 \).

For this end we define \( h := f(\xi_0) - p(\xi_0) = f(0.5) - p(0.5) \) and set up a linear equation system for \( a_0, a_1, a_2 \) and \( h \) in such a way that the condition from Theorem 7.4 is fullfilled:

\[
\begin{align*}
\sin(0.5) - a_2 0.5^2 - a_1 0.5 - a_0 &= h \\
\sin(1.0) - a_2 - a_1 - a_0 &= -h \\
\sin(1.5) - a_2 1.5^2 - a_1 1.5 - a_0 &= h \\
\sin(2.0) - a_2 2.0^2 - a_1 2.0 - a_0 &= -h
\end{align*}
\]
7.11: Example

Points where the sign condition is applied (see Th. 7.4)

Levelled reference error $h$: $l = 4.8 \times 10^{-3}$
A point $\xi \in (a, b)$ with $f(\xi) = 0$ is called a double zero if $f$ does not change its sign at $\xi$. (Note, this definition calls zeros which classically are considered to have a higher but even multiplicity as “double” zeros.)

In the sequel we assume the existence of two best solutions $p^*$ and $q^*$ and set $r = q^* - p^*$. A uniqueness proof has to show that $r \equiv 0$.

**Theorem. [cf. Th. 7.5]**

Let $r \in C[a, b]$ and let $\mathcal{Z} = \{\xi_i, i = 0, \ldots, n + 1\}$ be a reference, with

$$(-1)^i r(\xi_i) \geq 0 \quad \forall \xi_i \in \mathcal{Z}.$$  

Then $r$ has at least $n + 1$ zeros in $[a, b]$ where double zeros are counted twice.
Theorem. [cf. Th. 7.6]
Let $A \subset C[a,b]$ be a Haar space. Then there exists for any $f \in C[a,b]$ a unique best minimax approximation in $A$.

Note, that the Haar condition is a necessary condition. There are examples for non Haar spaces which give multiple best approximations.
7.13: A lower bound

Let a function $p^*$ share the sign alternation property with the best approximation. Can we deduce from that a statement about the quality of the best approximation?

**Theorem. [cf. Th. 7.7]**

Let $\mathcal{A}$ be an $(n + 1)$ dimensional Haar space which fullfills the condition of the Characterization Theorem. For $p^* \in \mathcal{A}$ with the sign property

$$\text{sign} \left( f(\xi_{i+1}) - p^*(\xi_{i+1}) \right) = -\text{sign} \left( f(\xi_i) - p^*(\xi_i) \right) \quad \forall \xi_i \in \mathcal{Z},$$

where $\mathcal{Z} = \{\xi_i, i = 0, \ldots, n + 1\}$ is a reference, we get:

$$\min_{i=0:n+1} |f(\xi_i) - p^*(\xi_i)| \leq \min_{p \in \mathcal{A}} \max_{i=0:n+1} |f(\xi_i) - p(\xi_i)| \leq \min_{p \in \mathcal{A}} \|f - p\|_\infty \leq \|f - p^*\|_\infty$$

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