5.7: Hermite interpolation

Hermite interpolation task:

For a given function $f \in C^m[a, b]$ and given points $\{x_i, i = 0, \ldots, m\}$ find a polynomial $p \in \mathcal{P}^n$ such that for given points

$$p^{(j)}(x_i) = f^{(j)}(x_i) \quad j = 0, \ldots, l_i \quad i = 0, \ldots, m \quad n + 1 = \sum_{i=0}^{m} (l_i + 1).$$
5.8: Hermite interpolation: Solvability and uniqueness

Theorem. [cf. Th. 5.4]
The Hermite interpolation task has a unique solution, provided that the \( x_i \) are distinct.

The proof makes use of the fact that the functions \( x^j, j = 0, \ldots, n \) form a basis of \( P \).
Then it suffices to show that 0 data implies 0 coefficients.
Note furthermore that \( c \prod_{i=0}^{m} (x - x_i)^{l_i} \) solves the interpolation task but is a polynomial of degree \( n + 1 \) unless \( c = 0 \).
5.9: Divided Differences: Extension to the Hermite case

We allow multiplicity of arguments corresponding to multiple input data at the same interpolation point \(x_i\):

\[
F[x_0, \ldots, x_0, x_1, \ldots, x_{i-1}, x_i, \ldots, x_i, x_{i+1} \ldots, x_n].
\]

\((l_0+1)-\text{times}\)

\((l_i+1)-\text{times}\)

In the recurrence relation division by zero occurs. Consequently the definition of the divided differences is in this case slightly altered:

\[
F[x_i, \ldots, x_i] := \frac{1}{l_i!} f^{(l_i)}(x_i)
\]

\((l_i+1)-\text{times}\)
5.10: Example

Let’s find the Hermite interpolation polynomial to the data

\[
\sin(0) = 0, \sin'(0) = 1, \sin''(0) = 0, \sin'''(0) = -1, \sin(\pi/2) = 1
\]

The divided difference scheme is

\[
\begin{array}{c|ccccc}
 k & 0 & 1 & 2 & 3 & 4 \\
 \hline
 0 & 0 & & & & \\
 \pi/2 & 1 & 2/\pi & -0.231335 & -0.147272 & 0.012346 \\
\end{array}
\]

\[
\begin{array}{ccccc}
 & 0 & 0 & 1 & 0 & -\frac{1}{3!} \\
\end{array}
\]

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5.11: Example

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6.0: Uniform Convergence of polynomials

**Definition.** A sequence of functions \( f_i \) converges uniformly to \( g \) if for all \( \epsilon > 0 \) there exists a \( k_0 \in \mathbb{N} \) such that for all \( x \in [a, b] \) and all \( k > k_0 \)

\[
|f_k(x) - g(x)| < \epsilon
\]

Note, that in this definition \( k_0 \) depends only on \( \epsilon \), not on \( x \).

Uniform convergence and convergence in the \( \| \cdot \|_\infty \)-norm is the same.

**Definition.** A sequence of functions \( f_i \) converges pointwise to \( g \) if for all \( \epsilon > 0 \) and for all \( x \in [a, b] \) there exists a \( k_0 \in \mathbb{N} \) such that for all \( k > k_0 \)

\[
|f_k(x) - g(x)| < \epsilon
\]

Here \( k_0 \) depends on \( \epsilon \) and \( x \).
6.1: Weierstrass Theorem

**Theorem 1. [cf. Th. 6.1]**
Let $f \in C[a, b]$.

For any $\varepsilon > 0$ exists an $n \in \mathbb{N}$ and a $p \in \mathcal{P}^n$ such that

$$\|f - p\|_\infty \leq \varepsilon$$

Note, that $p$ needs not to be an interpolation polynomial. Making the interpolation grid denser and denser will not necessarily lead to the desired approximating polynomial.
6.2: Monotone operators

Definition. 
An operator $L : C[a, b] \rightarrow C[a, b]$ is called monoton if for all $f, g \in B$

\[ f(x) \geq g(x) \implies (Lf)(x) \geq (Lg)(x) \quad \forall x \in [a, b] \]

Evidently, if $L$ is linear, monotonocity is equivalent with

\[ f(x) \geq 0 \implies (Lf)(x) \geq 0 \quad \forall x \in [a, b] \]
6.3: Monotone Operator Theorem

**Theorem.** Let $L_i : C[a, b] \to C[a, b]$ be a sequence of linear monotone operators. Then,

$$
\lim_{i \to \infty} L_ip = p \quad \forall p \in \mathcal{P}^2 \Rightarrow
$$

$$
\lim_{i \to \infty} L_if = f \quad \forall f \in C[a, b]
$$

The proof shows first pointwise convergence according to the figure. Then uniform convergence is proved.
6.4: Bernstein approximation operator

We consider for a while the interval \([a, b] = [0, 1]\):

**Definition.**

*The linear operator* \(B_n : C[0, 1] \to \mathbb{P}^n\) *with*

\[
(B_n f)(x) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k (1-x)^{(n-k)} f(k/n)
\]

is called *Bernstein operator of degree* \(n\).

Note, that \(B_n\) is not a projection.

**Theorem.** [cf. Th. 6.3]

\[
\lim_{n \to \infty} B_n f = f \quad \forall f \in C[0, 1]
\]
6.5: Repeated application of the Bernstein operator

The figure shows the result of $B_n^k f$, $k = 0, \ldots, 100$ for $f = \sin$. Besides that it demonstrates that $B_n$ is no projection, it shows also that the straight line is a fixed point of $B_n$. 
6.6: Derivative of Bernstein approximation

The Bernstein operator gives even uniform convergence for the derivative:

**Theorem. [cf. Th. 6.4]**

Let \( f \in C^1[0, 1] \), then

\[
\lim_{n \to \infty} \| f' - (B_nf)' \|_\infty = 0
\]