Numerical Approximation

Carmen Arévalo Claus Führer

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4.3: Interpolation Error

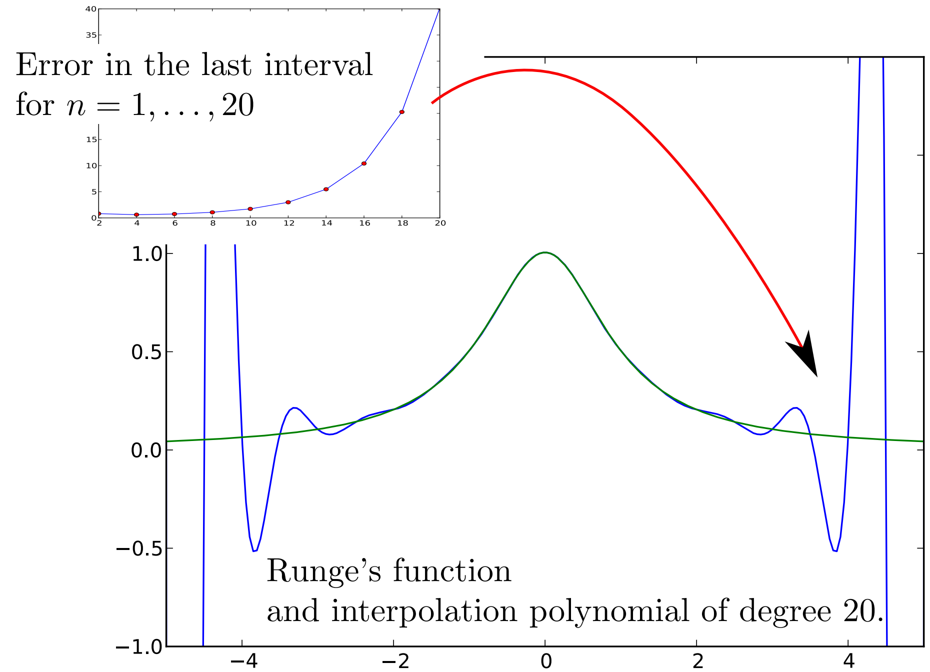
Theorem. [cf. Th. 4.2] *Let* $x_i, i = 0, \ldots, n \subset [a, b]$ *be distinct points,* $f \in \mathcal{C}^{(n+1)}[a, b]$ *and* $p \in \mathcal{P}^n$ *the unique polynomial which interpolates* f *at the* x_i*.*
Then, for all $x \in [a, b]$ *there exists a* $\xi \in [a, b]$*, such that*

$$e(x) = f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \Pi_{i=0}^{n} (x - x_i)$$

Proof: based on subsequent application of Rolle's theorem.

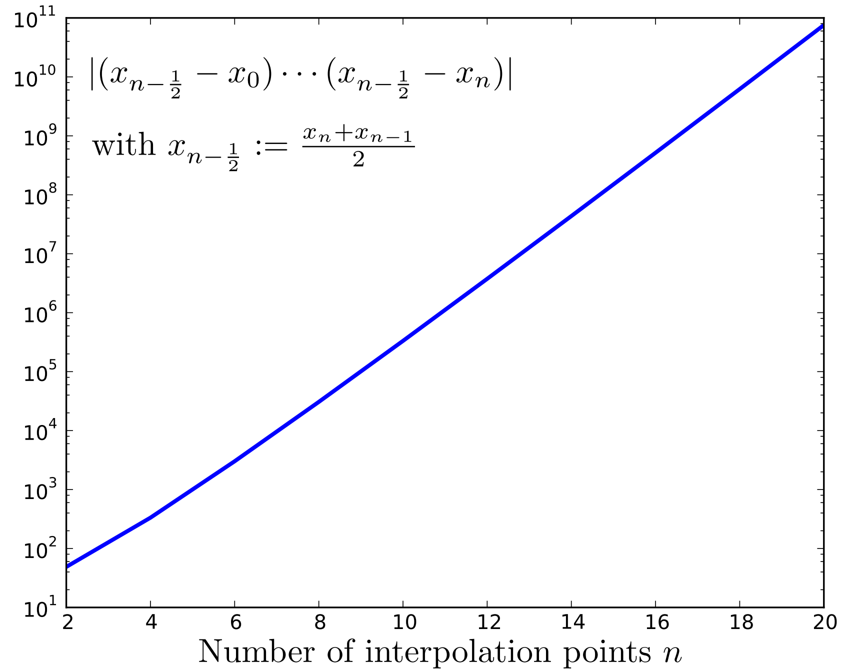
4.4: Runge's Example

By interpolating the function $f(x) = \frac{1}{1+x^2}$ on an equidistant grid, with increasing number of grid points one can demonstrate the disadvantages of high degree polynomial interpolation on equidistant grids.



4.5: Interpretation of the error formula applied to Runge's Example

Growth of $\Pi_{i=0}^{n}(x - x_i)$, for x being the midpoint in the last interval.



4.6: Chebyshev polynomials

Let $x \in [a,b] = [-1,1]$. And define $T_n(x) := \cos(n\theta)$ with $\theta = \arccos x$.
Note:

$$\cos[(n+1)\theta] + cos[(n-1)\theta] = 2\cos\theta\cos(n\theta)$$

which gives a 3-term recursion formula for $T_n(x)$:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

which is started by $T_0(x) \equiv 1$ and $T_1(x) = x$.

Thus, $T_n \in \mathcal{P}_n$. They are called the Chebyshev polynomials of degree n.

4.7: Chebyshev Polynomials – Properties

- The T_i have integer coefficients.
- The leading coefficient is $a_n = 2^{n-1}$.
- T_{2n} is even, T_{2n+1} is odd.
- $|T_n(x)| \leq 1$ for $x \in [-1, 1]$ and $|T_n(x)| = 1$ for $x_k := \cos(k\pi/n)$.
- $T_n(1) = 1, T_n(-1) = (-1)^n$
- $T_n(\bar{x}_k) = 0$ for $\bar{x}_k = \cos\left(\frac{2k-1}{2n}\pi\right)$ for $k = 1, \ldots, n$ (so-called Chebyshev points)

4.8: Scaling of $[-1, 1]$

In case of $[a, b] \neq [-1, 1]$ we have to consider the map:

$$[a, b] \to [-1, 1] \qquad x \mapsto 2\,\frac{x-a}{b-a} - 1$$

4.10: Norm of Interpolation operator

Theorem. [cf. Th. 4.3]
The norm of the interpolation operator is

$$\|X\| = \max_{x\in[a,b]} \sum_{k=0} |l_k(x)|.$$