# Numerical Approximation
# Slides and Course Summary

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## 0: Preamble

These slides form the skeleton of the course. They have to be completed by blackboard notes and comments to the exercises.

The notation and enumeration of theorems and definitions and references to pages and equations follow the course book
M.J.D. Powell: Approximation Theory and Methods

## 1: The Problem

Given sets $\mathcal{B}$ and $\mathcal{A}$ and some element $f \in \mathcal{B}$.

Tasks:

- Find an element $a \in \mathcal{A}$ which (in some sense) best approximates $f$.
- Make a statement about the quality of the approximation.

Often $\mathcal{B}$ is a linear function space and $f$ is a function or
$\mathcal{B}$ is $\mathbb{R}^n$ and $f$ is a set of given data.

$\mathcal{A}$ is a subset or often a subspace of $\mathcal{B}$.

## 1.1: Examples

1. Given a set of points that are the solution to a differential equation, approximate the solution by a polynomial of fixed degree. How good the approximation is can be measured by substituting the polynomial into the differential equation and looking at the residual.

2. Approximate a complex (difficult) function by a rational function.

3. Fit a straight line to a set of n data points. In this case, $\mathcal{B}$ is $\mathcal{R}^n$ and the goodness of the approximation is measured as $|p(x_i) - y_i|$.

## 1.2: Distance, Norms

Let $\mathcal{B}$ be a linear space with a norm (normed linear space)

**Definition.** *A map $\|.\| : \mathcal{B} \to \mathbb{R}$ is called a norm iff for all $x, y \in \mathcal{B}$:*

- $\|x\| \geq 0$ *and* $\|x\| = 0 \Rightarrow x = 0$
- $\|\alpha x\| = |\alpha|\|x\|$ *for all* $\alpha \in \mathbb{R}$.
- $\|x + y\| \leq \|x\| + \|y\|$

The distance between two elements of $\mathcal{B}$ is

$$\mathrm{d}(x, y) = \|x - y\|$$

## 1.3: Best Approximations

**Definition.** *A set S in a normed space is compact if every sequence in S has a limit point in S.*

**Theorem. [cf. Th. 1.1]** *Let $\mathcal{A}$ be a compact subset of the normed space $\mathcal{B}$ and $f \in \mathcal{B}$.*
*There exists an $a^* \in \mathcal{A}$ such that*

$$\mathrm{d}(a^*, f) \leq \mathrm{d}(a, f) \quad \forall a \in \mathcal{A}$$

*$a^*$ is called the best approximation of $f$ in $\mathcal{A}$.*

If $\mathcal{A}$ is a finite-dimensional subspace of $\mathcal{B}$, then compact is equivalent to closed and bounded.

**Theorem. [cf. Th. 1.2]** *Let $\mathcal{A}$ be a finite dimensional subspace of $\mathcal{B}$ and $f \in \mathcal{B}$.*
*There exists a best approximation of $f$ in $\mathcal{A}$.*

Often, $\mathcal{B} = \mathbb{R}^n$ or $\mathcal{B} = C[a, b]$

# 1.4: Norms in $\mathbb{R}^n$ and $C(a,b)$

The best approximation depends on the choice of a norm (distance function).

Common norms in $\mathbb{R}^n$ are

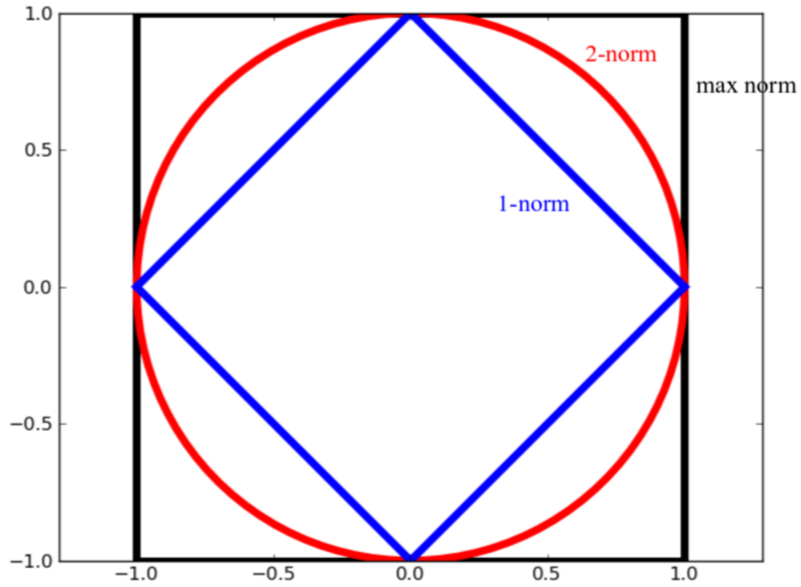
- $\|x\|_1 = \sum_{i=1}^n |x_i|$
- $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$
- $\|x\|_\infty = \max_{i=1:n} |x_i|$

Norms in $C([a,b])$ are the $L_p$ norms

- $\|f\|_1 = \int_a^b |f(x)| \mathrm{d}x$
- $\|f\|_2 = \sqrt{\int_a^b f(x)^2 \mathrm{d}x}$
- $\|f\|_\infty = \max_{x \in [a,b]} |f(x)|$

# 1.5: Unit circle in $\mathbb{R}^2$

The unit circle is the set of all elements which have norm one.
Here the unit circle in $\mathbb{R}^n$ for different norms:



## 1.6: The $\infty$-norm as upper bound

**Theorem. [cf. Th. 1.3]**

$$\|e\|_1 \leq (b-a)^{1/2}\|e\|_2 \leq (b-a)\|e\|_\infty \quad \forall e \in \mathcal{C}[a,b]$$

Interpretation:

When $\|e\|_\infty$ tends to zero, then $e$ tends to zero also in the two other norms.

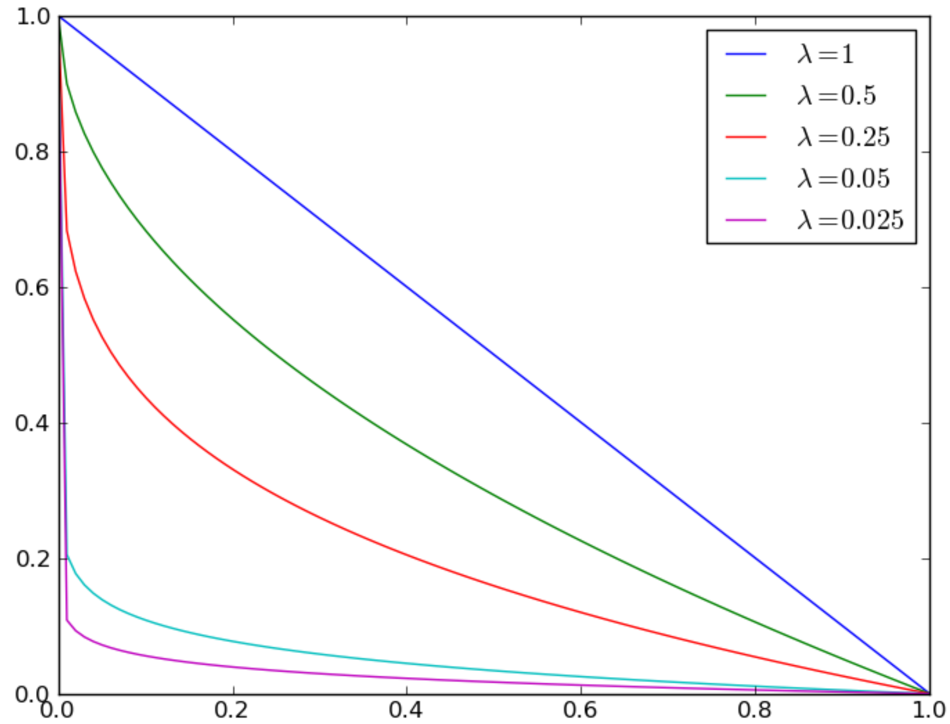
# 1.7: A counterexample

Let $e = 1 - x^\lambda$ with $\lambda \geq 0$.

$\lim_{\lambda \to 0} \|e\|_1 = 0$,
$\lim_{\lambda \to 0} \|e\|_2 = 0$,

but

$\lim_{\lambda \to 0} \|e\|_\infty = 1$



# 1.8: The geometric aspect of a best approximation

A ball of radius $r$ around $f \in \mathcal{B}$ is the set:
$\mathcal{N}(f,r) := \{g : \|f-g\| \leq r; g \in \mathcal{B}\}$.

Note:
$r_1 < r_2 \Rightarrow \mathcal{N}(f,r_1) \subset \mathcal{N}(f,r_2)$.