Numerical Approximation
The exchange algorithm

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8.0: Exchange Algorithm – Idea

- We consider an \( n + 1 \)-dimensional Haar space \( \mathcal{A} \subset C[a, b] \).

- The exchange algorithm is an iterative procedure to compute for a given \( f \) an arbitrarily good approximation to the best approximation of \( f \) in \( \mathcal{A} \).

- It starts with a reference \( \mathcal{Z} = \{ \xi_i, i = 0, \ldots, n + 1 \} \) and constructs a sequence of references which converges to a reference \( \mathcal{Z}_\mathcal{M} = \{ \xi^*_i, i = 0, \ldots, n + 1 \} \) which fulfills the conditions of the Characterization Theorem.
8.1: Steps of the exchange algorithm

<table>
<thead>
<tr>
<th>Input: basis functions $\varphi_i$, dimension $dim$, function $f(x)$ and its range $[a, b]$, reference $ref$ with $dim + 1$ distinct points.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Compute best approximation on the reference.</td>
</tr>
<tr>
<td>Compute reference level $h$.</td>
</tr>
<tr>
<td>Initialize $error_bnd = 1$.</td>
</tr>
<tr>
<td>while $error_bnd &gt; Tol$:</td>
</tr>
<tr>
<td>exchange one reference point</td>
</tr>
<tr>
<td>Compute best approx. on new reference</td>
</tr>
<tr>
<td>compute new $error_bnd$</td>
</tr>
<tr>
<td>Increase iteration counter $iter$</td>
</tr>
<tr>
<td>$iter &gt; max_iter$</td>
</tr>
<tr>
<td>break</td>
</tr>
</tbody>
</table>

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8.2: Best approximation on a reference

Let $\phi_i, \ i = 1, \ldots, \dim$ be a basis of $\mathcal{A}$ and let $p_Z(x) = \sum c_i \phi_i(x)$ be the best approximation of $f$ on $Z$. The coefficients $c_i$ are obtained from:

$$
\begin{pmatrix}
\phi_0(\xi_0) & \phi_0(\xi_1) & \cdots & \phi_0(\xi_n) & 1 \\
\phi_1(\xi_0) & \phi_1(\xi_1) & \cdots & \phi_1(\xi_n) & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_n(\xi_0) & \phi_n(\xi_1) & \cdots & \phi_n(\xi_n) & (-1)^{n-1}
\end{pmatrix}
\begin{pmatrix}
c_0 \\
c_1 \\
\vdots \\
c_n \\
h
\end{pmatrix}
= 
\begin{pmatrix}
f(\xi_0) \\
f(\xi_1) \\
\vdots \\
f(\xi_n)
\end{pmatrix}
$$

From this the reference level error $h$ is obtained and an error function $e(x) = f(x) - p_Z(x)$ can be defined.
8.3: Exchange a reference point

• By sampling $e(x)$ obtain $\xi^*$, with $|e_{\text{max}}| = |e(\xi^*)| = \max |e(x)|$.

• Find two neighboring points $\xi_i < \xi^* < \xi_{i+1}$

• If $\text{sign} e(\xi_i) = \text{sign} e_{\text{max}}$, then set $\xi_i := \xi^*$ else $\xi_{i+1} := \xi^*$

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8.4: Error bound

\[ |h| \leq \|f - p^*\|_\infty \leq \|f - p_Z\|_\infty \]

Computed obtained from sampling \(e(x)\)
8.5: Example 1

\[ f(x) = \sin(x) \quad [a, b] = [0, \pi/2] \quad \mathcal{A} = \mathcal{P}^2 \quad \mathcal{Z}_0 = \text{linspace}(0, \pi/2, 4) \]

Difference to best approximation less then Tol = $10^{-10}$ reached after 3 iterations.

\[ p_2(x) = -0.33142935 x^2 + 1.17488113 x - 0.01386491 \]

\[ e(x) = \sin(x) - p_2(x) \]

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8.6: Example 2

\[ f(x) = \sin(x) \quad [a, b] = [0, \pi/2] \quad \mathcal{A} = \text{span}\{x, x^2\} \quad \mathcal{Z}_0 = \{0.5, 1, \pi/2\} \]

Difference to best approximation less then \( \text{Tol} = 10^{-10} \) reached after 4 iterations.

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8.7: Example 2

Statistics:

<table>
<thead>
<tr>
<th>iteration</th>
<th>$\xi_0$</th>
<th>$\xi_1$</th>
<th>$\xi_2$</th>
<th>error bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.29904940</td>
<td>1.00000000</td>
<td>$\pi/2$</td>
<td>$2.7 \times 10^{-3}$</td>
</tr>
<tr>
<td>2</td>
<td>0.29904940</td>
<td>1.10490884</td>
<td>$\pi/2$</td>
<td>$9.0 \times 10^{-4}$</td>
</tr>
<tr>
<td>3</td>
<td>0.28330996</td>
<td>1.10490884</td>
<td>$\pi/2$</td>
<td>$2.0 \times 10^{-5}$</td>
</tr>
<tr>
<td>4</td>
<td>0.28330996</td>
<td>1.10490884</td>
<td>$\pi/2$</td>
<td>$1.6 \times 10^{-16}$</td>
</tr>
</tbody>
</table>
Theorem. [cf. Th. 8.1]
Let \( f \in P^{n+1} \subset C[-1, 1] \) and \( \mathcal{Z} = \{ \cos \left( \frac{n+1-i}{n+1} \pi \right) : i = 0, \ldots, n+1 \} \) be a reference consisting of Chebychev points only, then \( p_{\mathcal{Z}} = p_\ast \), i.e. the best approximation of \( f \) on the reference \( \mathcal{Z} \) is the best approximation on the entire interval \([-1, 1]\) and the exchange algorithm converges in one step.

The importance of this statements becomes clear, when considering a Taylor expansion of \( f \):

\[
f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \ldots + \frac{1}{(n+1)!}f^{(n+1)}(0)x^{n+1} + \frac{1}{(n+2)!}f^{(n+2)}(0)x^{n+2} + \ldots
\]

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8.9 Discrete Exchange Algorithm

Often $f$ is not given as a function but as a set of function values on a grid

$$\mathcal{G} = \{x_1, x_2, \ldots, x_m\} \subset [a, b] \quad m > \dim(\mathcal{A}).$$

The discrete version of the exchange algorithm is a simple modification, with $\mathcal{Z} \subset \mathcal{G}$.

**Theorem.** [cf. Th. 8.2]

*The discrete exchange algorithm converges in a finite number of steps.*
9.0 Convergence of the Exchange Algorithm

We denote the reference at step $k$ of the algorithm by $Z_k = \{\xi_0^{(k)}, \ldots, \xi_{n+1}^{(k)}\}$ and the corresponding polynomial by $p^{(k)}$.

The proof shows that the values $h(\xi_0^{(k)}, \ldots, \xi_{n+1}^{(k)}):= |h^{(k)}|$ are monotonically increasing with $k$.

As these values are bounded by $\|f - p^*\|_\infty$ convergence of the $|h^{(k)}|$ follows, which by a simple argument implies $\lim_{k \to \infty} p^{(k)} = p^*$.

The general assumption is that $\mathcal{A} \subset C[a, b]$ is an $n + 1$-dimensional Haar space.
9.1: Eliminating $p^{(k)}$ \hfill (1)

A first step in the convergence proof is the elimination of

$$p^{(k)} = \sum_{j=0}^{n} \lambda_j^{(k)} \phi_j$$

$$f(\xi_i^{(k)}) - \sum_{j=0}^{n} \lambda_j^{(k)} \phi_j(\xi_i^{(k)}) = (-1)^i h^{(k)} \quad i = 0, \ldots, n + 1$$

There exists $\sigma_i$ (not all 0) such that

$$\sum_{i=0}^{n+1} \sigma_i \phi_j(\xi_i^{(k)}) = 0 \quad j = 0, \ldots, n$$
9.2: Eliminating $p^{(k)}$ (2)

and we get

$$\sum_{i=0}^{n+1} \sigma_i f(\xi_i^{(k)}) = \sum_{i=0}^{n+1} \sigma_i (-1)^i h^{(k)}$$

**Theorem.** [cf. Th. 9.1]

Let $\{\sigma_i, i = 0, \ldots, n + 1\}$ be a set of real numbers, which are not all 0 and for which

$$\sum_{i=0}^{n+1} \sigma_i p(\xi_i) = 0 \quad \forall p \in A$$

holds, then

$$\sigma_i \neq 0 \quad \sigma_i = -\sigma_{i+1}$$