

Numerical Methods in CAGD

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AFFINE SPACES

Let \mathcal{A} be a nonempty set of **points**.

Define the map

$$\mathbf{v} = p - q$$

such that

- for any $p, q \in \mathcal{A}$ there is a unique $\mathbf{v} \in \mathbb{R}^3$ such that $p = q + \mathbf{v}$;
- $p + (\mathbf{v} + \mathbf{w}) = (p + \mathbf{v}) + \mathbf{w}$ for any $p \in \mathcal{A}$ and any $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$

\mathcal{A} with its underlying vector space is called an **affine space**.

(Note that the sum of two points is not defined)

BARYCENTRIC (affine) COMBINATIONS

Suppose $\dim(\mathcal{A}) = n$ and points p_0, p_1, \dots, p_n are such that the vector sequence $\mathbf{v}_1 = p_1 - p_0, \mathbf{v}_2 = p_2 - p_0, \dots, \mathbf{v}_n = p_n - p_0$ is linearly independent.

Then any point $q \in \mathcal{A}$ can be written as

$$\begin{aligned} q &= p_0 + \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n \\ &= \alpha_0 p_0 + \alpha_1 p_1 + \dots + \alpha_n p_n \end{aligned}$$

where $\alpha_0 + \alpha_1 + \dots + \alpha_n = 1$.

The α_i are the **barycentric coordinates** of q with respect to p_0, \dots, p_n .

CONVEX COMBINATIONS and CONVEX HULL

If the affine combination $q = \sum_{i=0}^n \alpha_i p_i$ $\left(\sum_{i=0}^n \alpha_i = 1 \right)$ satisfies $\alpha_i \geq 0$, it is called a **convex combination**.

On the other hand, $\sum_{i=0}^n \alpha_i p_i$ $\left(\sum_{i=0}^n \alpha_i = 0 \right)$ generates a vector:

$$\mathbf{v} = \alpha_1(p_1 - p_0) + \cdots + \alpha_n(p_n - p_0)$$

The **convex hull** of points p_0, \dots, p_n is the smallest subset of \mathcal{A} containing all the line segments between the points, which may be written as

$$q = (1 - \alpha)p_i + \alpha p_j, 0 \leq \alpha \leq 1$$

Convex combinations lie in the convex hull of the points p_0, \dots, p_n .

AFFINE MAPS

Let \mathcal{A} and \mathcal{B} be affine spaces of dimensions m and n resp.

$\Phi : \mathcal{A} \rightarrow \mathcal{B}$ is called an **affine map** if it preserves barycentric combinations:

$$\Phi \left(\sum \alpha_i p_i \right) = \sum \alpha_i \Phi(p_i) \quad \text{when} \quad \sum \alpha_i = 1$$

When points and vectors are described by its coordinates in a given coordinate system, affine maps can be represented by an $n \times m$ matrix A and a vector \mathbf{v} such that

$$\Phi(x) = Ax + \mathbf{v}$$

An affine map consists of a linear transformation followed by a translation.

PARAMETRIC CURVES and SURFACES

We may define a vector in space as having coordinates that depend on a parameter t ,

$$\mathbf{v}(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{pmatrix}$$

As t varies, $\mathbf{v}(t)$ traces a **parametric curve**. If the coordinate functions are polynomials of degree $\leq n$, we call it a **polynomial curve** of degree n in t .

If there are two parameters, it traces a **parametric surface** resp. a **polynomial surface** of degree n in s, t (if all coordinate bivariate functions $v_i(s, t)$ are polynomials of total degree n in s and t).