Conservation Laws & Finite Volume Methods

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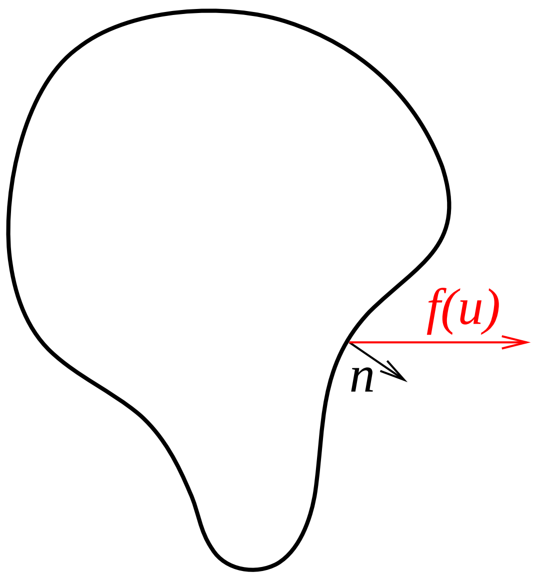
conservation laws

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(t,x)\,\mathrm{d}x + \int_{\partial\Omega} f(u(t,x))\cdot n\,\mathrm{d}S = 0$$



$$\boxed{u_t + \nabla\cdot f(u) = 0}$$

$$|u-k|_t + (\mathsf{sign}(u-k)(f(u)-f(k)))_x \leq 0\ , \quad \forall k\in\mathbb{R}$$

$$\int_0^\infty\int_{\mathbb{R}} |u-k|\phi_t + \mathsf{sign}(u-k)(f(u)-f(k))\phi_x \mathrm{d}x\mathrm{d}t \geq 0\ , \quad \forall k,\phi\ldots$$

conservation laws

shallow water

$$\begin{pmatrix} h \\ hu \\ hv \end{pmatrix}_t + \begin{pmatrix} hu \\ hu^2 + h^2/2 \\ huv \end{pmatrix}_x + \begin{pmatrix} hv \\ huv \\ hv^2 + h^2/2 \end{pmatrix}_y = 0$$

conservation laws

Euler equations

$$\begin{pmatrix} \rho \\ m \\ n \\ E \end{pmatrix}_t + \begin{pmatrix} m \\ um + p \\ un \\ u(E+p) \end{pmatrix}_x + \begin{pmatrix} n \\ vm \\ vn + p \\ v(E+p) \end{pmatrix}_y = 0$$

$$\begin{aligned} u &= m/\rho \\ v &= n/\rho \\ p &= (\gamma - 1)\left[E - \frac{m^2+n^2}{2\rho}\right] \end{aligned}$$

conservation laws

applications

- ▷ acoustic waves in the atmosphere, the ocean, or solids
- ▷ shock waves and rarefaction waves in gas dynamics
- ▷ electromagnetic waves, visible light, radar
- ▷ shallow water waves
- ▷ ultrasound waves
- ▷ traffic dynamics
- ▷ porous media flow, (oil) reservoirs, blood flow
- ▷ waves arising from chemical reactions
- ▷ combustion of gases, detonation, deflagration
- ▷ waves in plasmas and ionized gases (MHD)
- ▷ gravitational waves, colliding black holes . . .

conservation laws

tentative outline (linear systems)

▷ examples:

- gas dynamics (2.6)
- linear accoustics (2.7)
- shallow water (13.1)

▷ linear hyperbolicity (3)

▷ finite–volume methods:

- first–order (4)
- high–resolution (6)

▷ stability, convergence, accuracy (8)

conservation laws

tentative outline (nonlinear problems)

- ▷ scalar nonlinear conservation laws:
  - traffic flow (11.1)
  - Burgers' equation (11.3)
- ▷ weak solutions: shocks, rarefaction waves, entropy (11)
- ▷ finite–volume methods for nonlinear problems (12)
- ▷ nonlinear systems:
  - shallow water (13)
  - gas dynamics (14)
- ▷ finite–volume methods for nonlinear systems (15)
- ▷ stability and convergence (16)
- ▷ relaxation and kinetic methods

conservation laws

isentropic gas

$$\begin{pmatrix} \rho \\ \rho u \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \end{pmatrix}_x = 0$$

$$p = \kappa \rho^\gamma \ , \quad (\text{air: } \gamma = 1.4)$$

polytropic gas

$$\begin{pmatrix} \rho \\ m \\ E \end{pmatrix}_t + \begin{pmatrix} m \\ um + p \\ u(E+p) \end{pmatrix}_x = 0$$

$$m = \rho u \ , \quad p = (\gamma - 1)\left(E - \frac{m^2}{2\rho}\right)$$

conservation laws

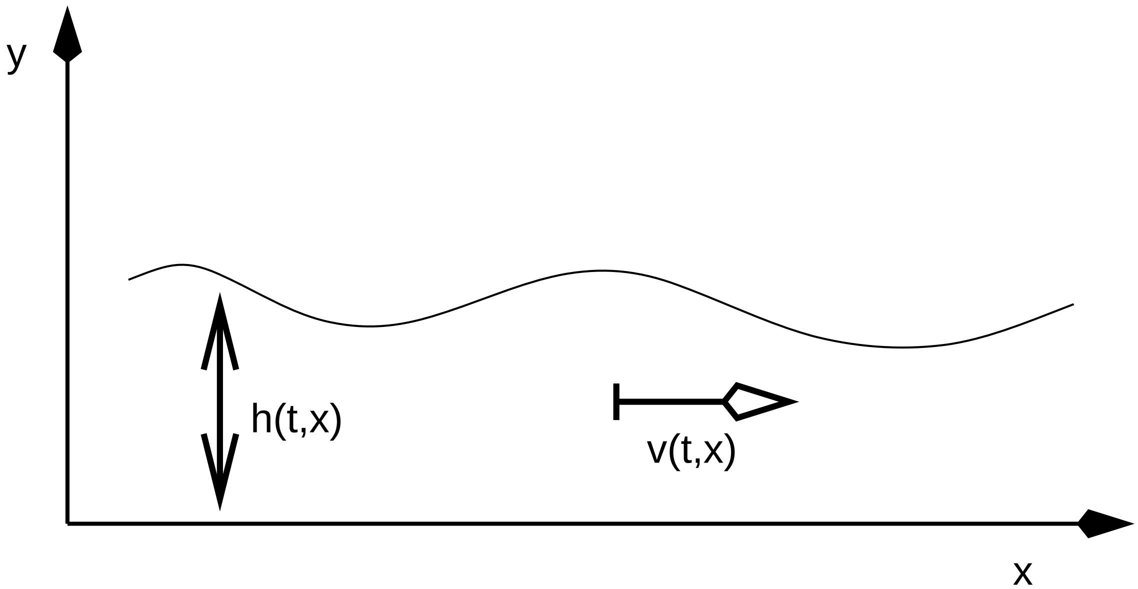
linear acoustics

$$\begin{pmatrix} p \\ u \end{pmatrix}_t + \begin{pmatrix} u_0 & K_0 \\ 1/\rho_0 & u_0 \end{pmatrix} \begin{pmatrix} p \\ u \end{pmatrix}_x = 0$$

$$K_0 = \rho_0 p'(\rho_0)$$

conservation laws

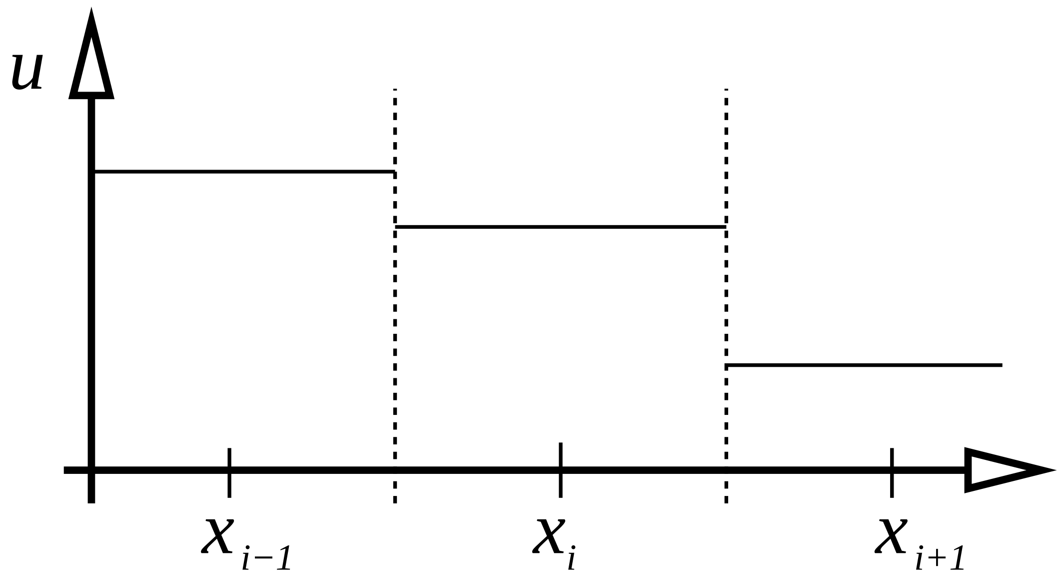
shallow water

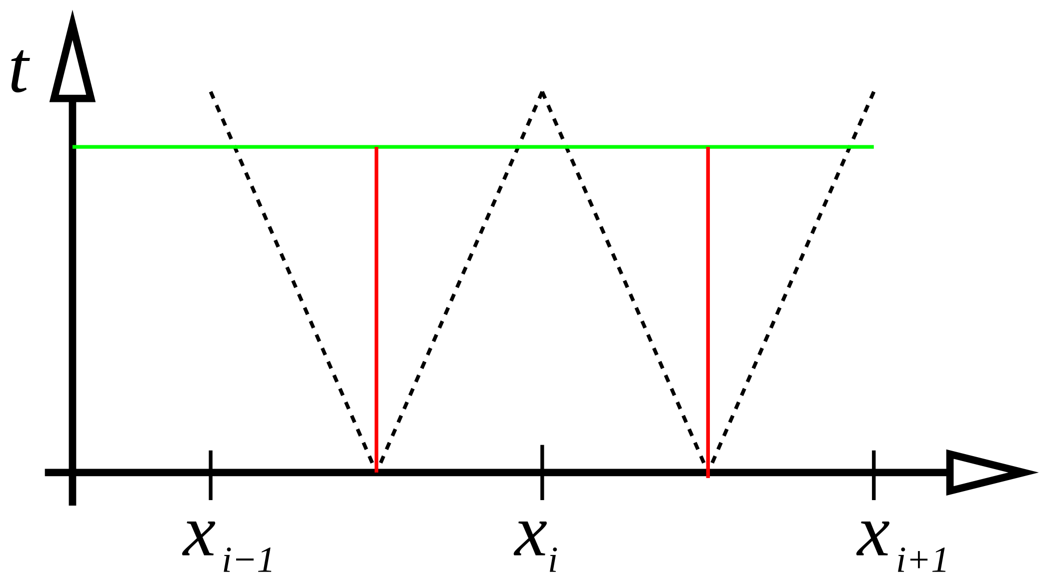


$$\begin{aligned} h_t &+ (vh)_x &= 0 \\ (vh)_t &+ (v^2h + gh^2/2)_x &= 0 \end{aligned}$$

Godunov's method (1959)

Riemann problem (1860)





CFL condition (1928): $\frac{\Delta t}{\Delta x} |\lambda|_{max} \leq \left(\frac{1}{2} \leq\right) 1$

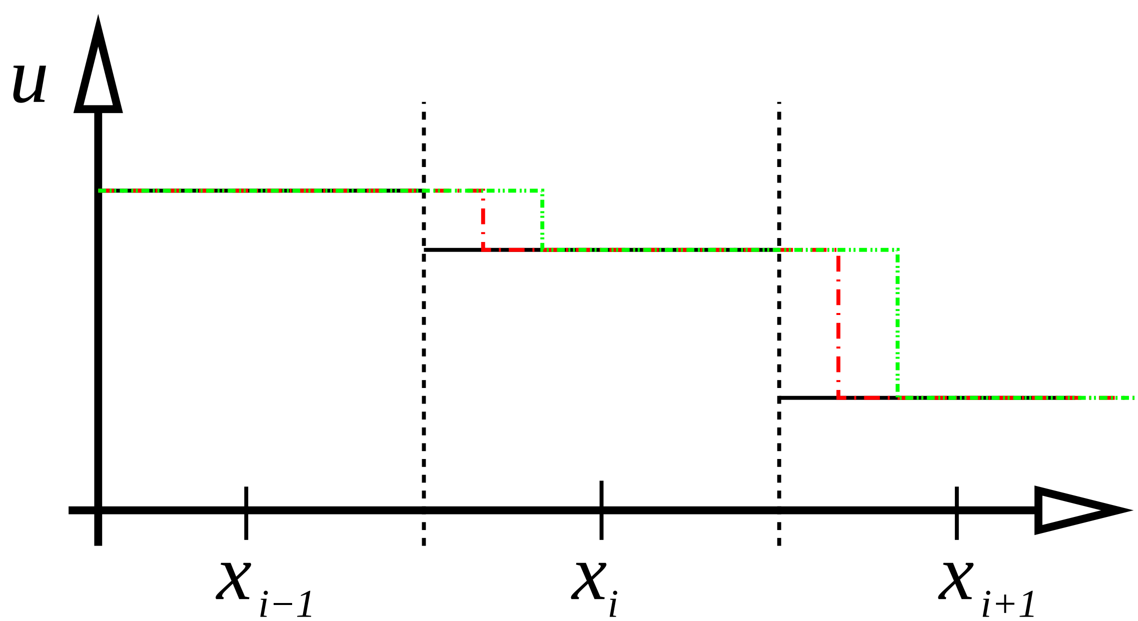
Godunov's method . . .

advection equation

$$u_t + au_x = 0 \ , \quad u(0,x) = u^0(x)$$

travelling wave solution

$$u(t,x) = u^0(x - at)$$



Godunov's method . . .

REA algorithm (1959):

▷ **Reconstruct** a piecewise constant function

$$\widetilde{q}(x, t_n) = Q_i^n \quad , \quad x \in \mathcal{C}_i$$

▷ **Evolve** the conservation law with this data

$$\widetilde{q}(x, t_n) \rightsquigarrow \widetilde{q}(x, t_{n+1})$$

▷ **Average** the solution at t_{n+1}

$$Q_i^{n+1} = \frac{1}{\Delta x} \int_{\mathcal{C}_i} \widetilde{q}(x, t_{n+1})\, \mathrm{d}x$$

High resolution method

▷ **Reconstruct** a piecewise linear function

$$\widetilde{q}(x, t_n) = Q_i^n + \sigma_i^n (x - x_i) \quad , \qquad x \in \mathcal{C}_i$$

such that $TV(q(\cdot, t_n)) \leq TV(Q^n)$

▷ **Evolve** the conservation law with this data

$$\widetilde{q}(x, t_n) \rightsquigarrow \widetilde{q}(x, t_{n+1})$$

scalar case: $TV(\widetilde{q}(\cdot, t_{n+1})) \leq TV(\widetilde{q}(\cdot, t_n))$

▷ **Average** the solution at t_{n+1}

$$Q_i^{n+1} = \frac{1}{\Delta x} \int_{\mathcal{C}_i} \widetilde{q}(x, t_{n+1}) \, \mathrm{d}x$$

$$TV(Q^{n+1}) \leq TV(\widetilde{q}(\cdot, t_{n+1}))$$

High resolution method

limited slopes: