2.13: How to compute matrix norms

Matrix norms are computed by applying the following formulas:

1-norm (Th. 2.8): \[ \| A \|_1 = \max_{j=1:n} \sum_{i=1}^{n} |a_{ij}| \] maximal column sum

∞-norm (Th. 2.7): \[ \| A \|_1 = \max_{i=1:n} \sum_{j=1}^{n} |a_{ij}| \] maximal row sum

2-norm (Th. 2.9): \[ \| A \|_2 = \max_{i=1:n} \sqrt{\lambda_i(A^T A)} \] where \( \lambda_i(A^T A) \) is the \( i^{th} \) eigenvalue of \( A^T A \).
2.14: Condition of a Problem

A mathematical problem can be viewed as a function mapping in-data to out-data (solution):\

$$f : D \subset \mathcal{V} \rightarrow \mathcal{W}$$

Condition number is a measure for the sensitivity of the out-data with respect to perturbations in in-data.

Scalar case: $y = f(x)$ and $\hat{y} = f(x + \delta x)$:

$$\hat{y} - y = f(x + \delta x) - f(x) = f'(x)\delta x + \frac{1}{2!}f''(x + \theta \delta x)(\delta x)^2$$
2.15: Condition of a Problem (Cont.)

Scalar case, relative:

\[
\hat{y} - y = \left( \frac{xf'(x)}{f(x)} \right) \frac{\delta x}{x} + O(\delta x^2)
\]

absolut (local) condition number

\[
\text{Cond}_x = |f'(x)|
\]

relative (local) condition number

\[
\text{cond}_x(f) = \left| \frac{xf'(x)}{f(x)} \right|
\]
2.16: Condition of a Problem (Cont.)

In general

\[ \text{absolut (local) condition number} \]
\[ \text{Cond}_x = \|f'(x)\| \]

\[ \text{relative (local) condition number} \]
\[ \text{cond}_x(f) = \frac{\|x\| \|f'(x)\|}{\|f(x)\|} \]

and we get

\[ \|\text{rel. output error}\| \leq \text{cond}_x(f)\|\text{rel. input error}\| \]
2.17: Examples

Example 1: (Summation)

Problem: \( f : \mathbb{R}^2 \longrightarrow \mathbb{R}^1 \) with \( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto x_1 + x_2 \)

Jacobian: \( f'(x_1, x_2)^T = (1, 1) \)

In 1-norm: \( \text{Cond}_{x_1, x_2}(f) = 1 \)

\[
\text{cond}_{x_1, x_2}(f) = \frac{|x_1| + |x_2|}{|x_1 + x_2|}
\]

Problem if two nearly identical numbers are subtracted.
Example 2: (Linear Systems)

Problem: \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) with \((b) \mapsto x = A^{-1}b\)

Jacobian: \( f'(b) = A^{-1} \)

In 1-norm: \( \text{Cond}_b(f) = \|A^{-1}\| \)

\[
\text{cond}_b(f) = \frac{\|b\|\|A^{-1}\|}{\|A^{-1}b\|} \leq \frac{\|A\|\|x\|\|A^{-1}\|}{\|x\|} = \|A\|\|A^{-1}\| =: \kappa(A)
\]
2.19: Examples (Cont.)

The estimate is sharp, i.e. there is a worst case perturbation

Example:

\[ A = \begin{pmatrix} 10^3 & 0 \\ 0 & 10^{-3} \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \delta b = \begin{pmatrix} 0 \\ 10^{-5} \end{pmatrix} \]

(see the exercises of the current week and Exercise 2.14 in the book.)
Unit 3: Systems of nonlinear functions

Example (p. 107)

\[ F(x) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} x_1^2 + x_2^2 - 1 \\ 5x_1^2 + 21x_2^2 - 9 \end{pmatrix} = 0 \]

\[ \xi_1 = (-\sqrt{3/2}, 1/2)^T \quad \xi_2 = (\sqrt{3/2}, 1/2)^T \]
\[ \xi_3 = (-\sqrt{3/2}, -1/2)^T \quad \xi_4 = (\sqrt{3/2}, -1/2)^T \]
3.1 Fixed Point Iteration in $\mathbb{R}^n$

**Definition. [4.1]** Let $g : D \subset \mathbb{R}^n \to \mathbb{R}^n$, $D$ closed (cf. p. 105), $x_0 \in D$ and $g(D) \subset D$. The iteration
\[ x_{k+1} = g(x_k) \]
is called fixed point iteration or simple iteration.

and similar to the scalar case we define

**Definition. [4.2]** Let $g : D \subset \mathbb{R}^n \to \mathbb{R}^n$, $D$ closed (cf. p. 105).
If there is a constant $L < 1$ such that
\[ \|g(x) - g(y)\| \leq L\|x - y\| \]
then $g$ is called contractive or contractive.
3.2 Contractivity and norms

Contractivity depends on the choice of a norm. Here a function, which is contractive in one norm, but not in another

\[ g(x) = \begin{pmatrix} 3/4 & 1/3 \\ 0 & 3/4 \end{pmatrix} x \]

It follows

\[ \|g(x) - g(y)\| = \|A(x - y)\| \leq \|A\|\|x - y\| \]

Thus \( L = \|A\| \).

But \( \|A\|_1 = \|A\|_\infty = \frac{13}{12} \) and \( \|A\|_2 = 0.9350 \).

\( g \) is contractive in the 2-norm and dissipative and the others.
3.3 Contractivity and norms (Cont.)

We start a fixed point iteration in the last example with $x = (1,1)^T$ and get the following norms:

(see also "equivalence of norms")
3.4 Fixed Point Theorem in $\mathbb{R}^n$

**Theorem. [4.1]** Let $g : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$, $D$ closed, $g(D) \subset D$. If there is a norm such that $g$ is contractive, then $g$ has a unique fixed point $\xi \in D$ and the fixed point iteration converges.

Let $J(x)$ be the Jacobian (functional matrix $\rightarrow$ flerdim) of $g$.

If $\|J(\xi)\| < 1$ then fixed point iterations converges in a neighborhood of $\xi$. (Th. 4.2)