2.6: Vector Norms

We generalize the definition of the absolute value of a real number to norms of vectors and matrices (later also of functions):

Definition. [2.6] *$\mathcal{V}$ a linear space , a function $\|\cdot\|: \mathcal{V} \to \mathbb{R}$ is called a norm if for all $u, v \in \mathcal{V}$ and all $\lambda \in \mathbb{R}$:*

- $\|v\| \geq 0$ *and* $\|v\| = 0 \Leftrightarrow v = 0$ *(Positivity)*
- $\|\lambda v\| = |\lambda| \|v\|$ *(Homogenity)*
- $\|u + v\| \leq \|u\| + \|v\|$ *(Triangular inequality)*

2.7: Examples

Examples for norms in $\mathbb{R}^n$:

- 1-norm

$$\|v\|_1 = \sum_{i=1}^{n} |v_i|$$

- 2-norm (Euclidean norm)

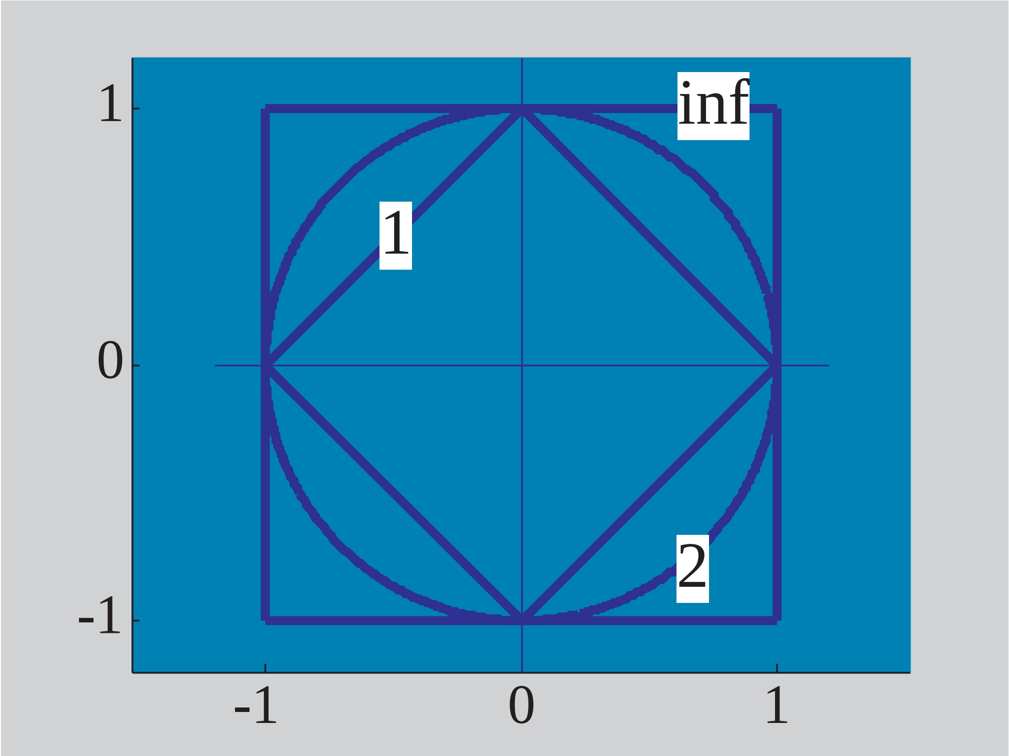
$$\|v\|_2 = \big(\sum_{i=1}^{n} v_i^2\big)^{1/2}$$

- ∞-norm

$$\max_{i=1:n} |v_i|$$

2.8: Unit Circle

The unit circle is the set of all vectors of norm 1.



2.10: Convergence

Theorem. [-]
If $\dim \mathcal{V} < \infty$ *and if* $\| \cdot \|_p$ *and* $\| \cdot \|_q$ *are norms in* $\mathcal{V}$*, then there exist constants* c, C *such that for all* $v \in \mathcal{V}$

$$c\|v\|_q \leq \|v\|_p \leq C\|v\|_q$$

Norms in finite dimensional spaces are equivalent.
Sequences convergent in one norm are convergent in all others.

2.11: Matrix norms

A matrix defines a linear map.

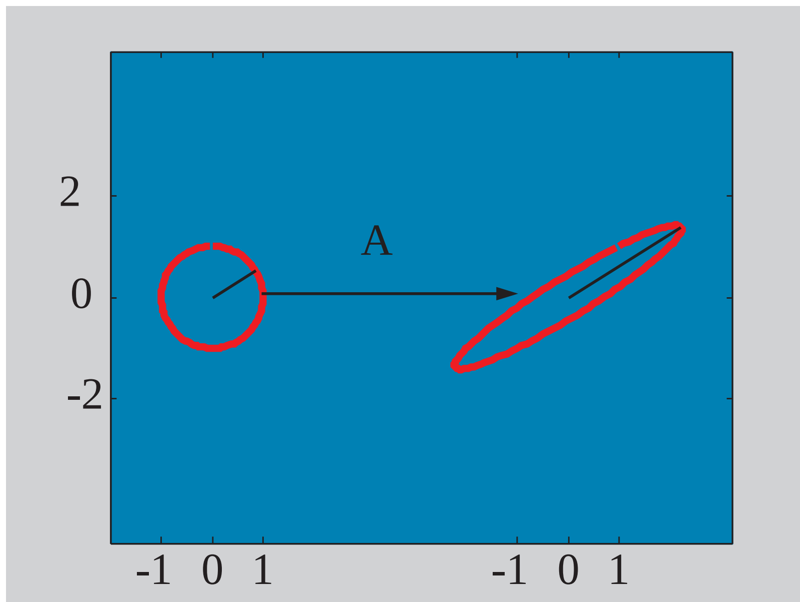
Definition. [2.10]
Let $\| \cdot \|$ be a given vector norm. The corresponding (subordinate) matrix norm is defined as

$$\|A\| = \max_{v \in \mathbb{R}^n \setminus \{0\}} \frac{\|Av\|}{\|v\|}$$

I.e. the largest relative change of a vector, when mapped by A.

2.12: Matrix norms

Example: $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$



$$\|A\| = 2.6180$$