8.12: Stability behavior of Euler’s method

We consider the so-called linear test equation

\[ \dot{y}(t) = \lambda y(t) \]

where \( \lambda \in \mathbb{C} \) is a system parameter which mimics the eigenvalues of linear systems of differential equations.

The equation is stable if \( \Re(\lambda) \leq 0 \). In this case the solution is exponentially decaying. \( (\lim_{t \to \infty} y(t) = 0) \).

When is the numerically solution \( u_i \) also decaying, \( \lim_{i \to \infty} u_i = 0 \)?
8.12: Stability behavior of Euler’s method (Cont.)

Explicit Euler discretization of linear test equation:

\[ u_{i+1} = u_i + h\lambda u_i \]

This gives \( u_{i+1} = (1 + h\lambda)^{i+1}u_0 \).

The solution is decaying (stable) if \(|1 + h\lambda| \leq 1\)
Implicit Euler discretization of linear test equation:

\[ u_{i+1} = u_i + h \lambda u_{i+1} \]

This gives \( u_{i+1} = \left( \frac{1}{1 - h \lambda} \right)^{i+1} u_0 \).

The solution is decaying (stable) if \( |1 - h \lambda| \geq 1 \)
8.14: Stability behavior of Euler’s method (Cont.)

Explicit Euler’s instability for fast decaying equations:

\[ \lambda = -5 \quad h = 0.41 \]
8.15: Stability behavior of Euler’s method (Cont.)

Facit:
For stable ODEs with a fast decaying solution ($\text{Real}(\lambda) \ll -1$) or highly oscillatory modes ($\text{Im}(\lambda) \gg 1$), the explicit Euler method demands small step sizes.

This makes the method inefficient for these so-called stiff systems.

Alternative: implicit Euler method.
8.16: Implementation of implicit methods

Implicit Euler method

\[ u_{i+1} = u_i + h_i f(t_{i+1}, u_{i+1}) \]

Two ways to solve for \( u_{i+1} \):

- **Fixed point iteration:**
  \[
  u_{i+1}^{(k+1)} = u_i + h_i f(t_{i+1}, u_{i+1}^{(k)}) = \varphi(u_{i+1}^{(k)})
  \]

- **Newton iteration:**
  \[
  u_{i+1} = u_i + h_i f(t_{i+1}, u_{i+1}) \Leftrightarrow \Delta u_{i+1} = 0
  \]
  \[
  F'(u_{i+1}^{(k)}) \Delta u_{i+1} = -F(u_{i+1}^{(k)})
  \]
  \[
  u_{i+1}^{(k+1)} = u_{i+1}^{(k)} + \Delta u_{i+1}
  \]

\( k \) is the iteration counter, \( i \) the integration step counter.
8.17: Implementation of implicit methods (Cont.)

These iterations are performed at every integration step! They are started with explicit Euler method as so-called predictor:

\[ u_{i+1}^{(0)} = u_i + h_i f(t_i, u_i) \]

When should fixed points iteration and when Newton iteration be used? The key is contractivity!

Let’s check the linear test equation again: \( \dot{y} = \lambda y \).

Contractivity: \( |\varphi'(u)| = |h\lambda| < 1 \).
8.18: Implementation of implicit methods (Cont.)

If the differential equation is

- nonstiff: explicit Euler or
- nonstiff: implicit Euler with fixed point iteration
- stiff: implicit Euler with Newton iteration