Unit 5: Cubic Splines

Let $K = \{x_0, \ldots, x_m\}$ be a set of given knots with

$$a = x_0 < x_1 < \cdots < x_m = b$$

**Definition. [11.2]** A function $s \in C^2[a, b]$ is called a cubic spline on $[a, b]$, if $s$ is a cubic polynomial $s_i$ in each interval $[x_i, x_{i+1}]$.

It is called a cubic interpolating spline if $s(x_i) = y_i$ for given values $y_i$. 
Interpolating cubic splines need two additional conditions to be uniquely defined

**Definition. [11.3]** An cubic interpolatory spline $s$ is called a natural spline if

$$s''(x_0) = s''(x_m) = 0$$
5.2: Cubic Splines - Construction

We construct an interpolating in a different but equivalent way than in the textbook:

Ansatz for $m$ the piecewise polynomials

$$s_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i$$

By fixing the $4m$ free coefficients $a_i, b_i, c_i, d_i, i = 0 : m - 1$ the entire spline is fixed.
5.3: Cubic Splines-Construction

We need $4m$ conditions to fix the coefficients

(1) $s_i(x_i) = y_i$, for $i = 0 : m - 1$,

(2) $s_{m-1} = y_m$, 1 condition

(3) $s_i(x_{i+1}) = s_{i+1}(x_{i+1})$, for $i = 0 : m - 2$,

(4) $s'_i(x_{i+1}) = s'_{i+1}(x_{i+1})$, for $i = 0 : m - 2$,

(5) $s''_i(x_{i+1}) = s''_{i+1}(x_{i+1})$, for $i = 0 : m - 2$,

These are $4m - 2$ conditions. We need two extra.
5.4: Cubic Splines-Boundary Conditions

We can define two extra boundary conditions. One has several alternatives:

**Natural Spline** \[ s''(x_0) = 0 \text{ and } s''_{m-1}(x_m) = 0 \]

**End Slope Spline** \[ s'(x_0) = y'_0 \text{ and } s'_{m-1}(x_m) = y'_m \]

**Periodic Spline** \[ s'(x_0) = s'_{m-1}(x_m) \text{ and } s''(x_0) = s''_{m-1}(x_m) \]

**Not-a-Knot Spline** \[ s'''(x_1) = s'''_{1}(x_1) \text{ and } s'''_{m-2}(x_{m-1}) = s'''_{m-1}(x_{m-1}) \]

We consider here natural splines.

MATLAB uses splines with a not-a-knot condition.
5.5: Natural Splines Construction

\[ s_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i \]

Let \( h = x_{i+1} - x_i \) equidistant spacing

\[ s_i'(t_{i+1}) = 3a_i h^2 + 2b_i h + c_i \]
\[ s_i''(t_{i+1}) = 6a_i h + 2b_i \]

From Condition (1) we get \( d_i = y_i \).

We introduce new variables for the second derivatives at \( x_i \), i.e.

\[ \sigma_i := s''(x_i) = 6a_i(x_i - x_i) + 2b_i = 2b_i \quad i = 0 : m \]
Thus \( b_i = \frac{\sigma_i}{2} \).

From
\[
\sigma_{i+1} = 6a_i h + 2b_i.
\]
and Condition (5) we get
\[
\sigma_{i+1} = 6a_i h + 2b_i.
\]

By Condition (3) we get
\[
y_{i+1} = a_i h^3 + b_i h^2 + c_i h + y_i.
\]
... and after inserting the highlighted expressions for \(a_i\) and \(b_i\) we get

\[
y_{i+1} = \left(\frac{\sigma_{i+1} - \sigma_i}{6h}\right)h^3 + \frac{\sigma_i}{2}h^2 + c_i h + y_i.
\]

From that we get \(c_i\):

\[
c_i = \frac{y_{i+1} - y_i}{h} - h\frac{2\sigma_i + \sigma_{i+1}}{6}.
\]

Using now Condition (4) gives a relation between \(c_i\) and \(c_{i+1}\)

\[
c_{i+1} = 3a_i h^2 + 2b_i h + c_i.
\]
5.8: Natural Splines Construction (Cont.)

Inserting now the expressions for $a_i$, $b_i$ and $c_i$, using Condition (2) and simplifying finally gives the central recursion formula:

$$
\sigma_{i-1} + 4\sigma_i + \sigma_{i+1} = 6\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right)
$$

with $i = 1, \ldots, m - 1$.

We consider now natural boundary conditions

$$
\sigma_0 = \sigma_m = 0.
$$
Finally we rewrite this all a system of linear equations

$$
\begin{pmatrix}
4 & 1 & & & \\
1 & 4 & 1 & & \\
1 & & & \ddots & 1 \\
& \ddots & & & 1 \\
1 & 4 & & & 1
\end{pmatrix}
\begin{pmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\vdots \\
\sigma_{m-1}
\end{pmatrix}
= \frac{6}{h^2}
\begin{pmatrix}
y_2 - 2y_1 + y_0 \\
y_3 - 2y_2 + y_1 \\
\vdots \\
y_n - 2y_{n-1} + y_{n-2}
\end{pmatrix}
$$

First, this system is solved and then the coefficients \( a_i, b_i, c_i, d_i \) are determined by the high-lighted equations.