Training Exam 2 in Numerical Analysis FMN050, 2017

The exam duration is 4 hours. In order to pass, a minimum of 15 points is required on the exam. The (preliminary) grade requirements are grade 3 ≥ 15 p, grade 4 ≥ 21 p and grade 5 ≥ 26 p.

You are allowed to use a pocket calculator, but no other material of any kind. Please answer the problems in Swedish or English.

Problems

1. Let $A$ be a $3 \times 3$-matrix with
   \[ \|A\|_2 = 2 \quad \text{and} \quad \|A^{-1}\|_2 = 3. \]
   How large can the solution’s relative error $\frac{\|\delta x\|_2}{\|x\|_2}$ be when solving $Ax = b$ with the data $b = (1.5, 0.21, 5.0)^T \pm (0.05, 0.005, 0.05)^T$? (2p)

2. We want to solve a linear system $Ax = b$ where the matrix $A$ has the following $LU$-factorization:
   \[
   A = LU = \begin{bmatrix}
   1 & 0 & 0 \\
   -1 & 1 & 0 \\
   0 & -1 & 1
   \end{bmatrix}
   \begin{bmatrix}
   2 & 1 & 0 \\
   0 & 1 & 3 \\
   0 & 0 & 2
   \end{bmatrix}
   \]
   a) Solve $Ax = b$, with $b = (1, 0, 3)^T$, by performing one forward and one backward substitution. (2p)
   
   b) What is gained by computing the $LU$-factorization of a matrix $A$ when solving a linear system $Ax = b$ instead of simply computing the inverse of $A$? (1p)

3. Consider two circles of radius one in the plane. The coordinates of the circles’ centers are $(0, 0)$ and $(0, 1)$, respectively. We would like to approximate the intersections of the two circles by employing Newton’s method.
   
   a) Reformulate the problem as a system of nonlinear equations with the form $f(x) = 0$. How many solutions are there? (1p)
   
   b) Construct the Jacobian matrix $f'(x)$. (1p)
   
   c) Introduce a suitable notation and write out Newton’s method for the problem at hand. (2p)
4. Construct the Newton interpolation polynomial that interpolates the data below. (3p)

\[
\begin{array}{c|ccc}
  x & -1 & 0 & 1 \\
  y & 1 & 0 & 2 \\
\end{array}
\]

5. Compute numerically the integral

\[
\int_{0}^{1} \frac{1}{1 + 25x^2} \, dx
\]

by employing the trapezoidal rule and dividing the interval \([0, 1]\) into \(N = 4\) equally sized intervals. (2p)

6. We want to solve the two-point boundary value problem

\[
y''(x) + y'(x) + \sin(x) y(x) = 0
\]

with boundary values \(y(0) = 1\) and \(y(1) = 1\). Introduce a suitable notation and discretize the problem by a standard second order method, i.e., the derivatives should be discretized as in Project 2. Construct the linear system (in matrix-vector form) that has to be solved, and make sure to include the boundary conditions. All details, such as matrix dimensions, vector lengths, grid, step size \(\Delta x\), etc., must be clearly stated. (5p)

7. Consider the following MATLAB code:

```matlab
N = 3;
a = 0; b = 1;
dx = (b-a)/N;
x = linspace(a+dx/2,b-dx/2,N)';
fx = cos(x);
weights = ones(1,N);
int = dx * weights * fx;
```

a) Which numerical method is implemented in the code, and what does the quantity \(\text{int}\) approximate? (2p)

b) Modify the code (without changing the problem being solved) so that the numerical approximation is improved. (1p)
8. Consider the initial value problem

\[ y'(t) = -y(t)^2 + \cos(t), \quad t > 0, \]

with the initial value \( y(0) = 2 \).

a) Approximate \( y(0.1) \) by using the explicit Euler method with the step size \( \Delta t = 0.1 \). (1p)

b) Explain what is meant by the statement: “The explicit Euler method has a convergence order of \( p = 1 \).” (1p)

c) Your friend has implemented a different method and solved the problem for various values of \( \Delta t \). The resulting global errors \( |y(0.1) - u_n| \) are given in the graph below, where \( u_n \) denotes the approximation of \( y(0.1) \). Estimate the convergence order \( p \) of your friend’s method. (2p)

![Graph showing convergence order](image)

9. There will be an additional problem, e.g., on the approximation errors of numerical integration/differentiation schemes, the fast Fourier transform or on the approximation of eigenvalues. (4p)

**LYCKA TILL!**