Exercises: Least squares and numerical integration

Numerical Analysis E3/I3, FMN050, 2017

Least squares method

1. We would like to fit a straight line $y(x) = c_0 + c_1 x$ to the data below by using the least squares method.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-1</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

a) Derive the overdetermined system $Ac \approx b$, which the coefficients $c = (c_0, c_1)^T$ need to fulfill.

b) Approximate a solution to the overdetermined system by the least squares method.

c) Plot the data points together with your straight line. Is the line a good fit to the data?

2. Consider the experimental measurements

<table>
<thead>
<tr>
<th>$x$</th>
<th>0.00</th>
<th>1.00</th>
<th>2.00</th>
<th>3.00</th>
<th>4.00</th>
<th>5.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>0.03</td>
<td>0.15</td>
<td>0.89</td>
<td>2.79</td>
<td>6.42</td>
<td>12.5</td>
</tr>
</tbody>
</table>

a) Derive the overdetermined system that a polynomial of degree $n$, with $0 \leq n \leq 5$, needs to satisfy in order to fit the data points.

b) Write a code that approximates a solution to the overdetermined system by the least squares method for $n = 0, 1, \ldots, 5$.

c) Plot the data points together with the computed polynomials. Which polynomial seems to capture the general trend in the data?

Numerical integration

3. Consider the following MATLAB code:

```matlab
N = 3 ;
a = 0 ;  b = 1 ;
dx = (b-a)/N ;
x = linspace(a+dx/2,b-dx/2,N)' ;
fx = cos(x) ;
weights = ones(1,N) ;
int = dx * weights * fx ;
```
a) Which numerical method is implemented in the code, and what does the quantity int approximate?

b) Modify the code (without changing the problem being solved) so that the numerical approximation is improved.

4. Consider the approximation of the integral

\[
\int_0^1 \frac{1}{1 + 25x^2} \, dx = 0.274680 \ldots
\]

a) Compute numerically the integral above by employing the trapezoidal rule and dividing the interval \([0, 1]\) into \(N = 4\) equally sized intervals.

b) Write a code for the trapezoidal approximation of the integral for a general \(N\). How large do you need to choose \(N\) in order to obtain an answer with six correct digits?

5. Compute numerically the integral

\[
\int_0^1 x^3 \, dx
\]

using the Trapezoidal rule with Richardson extrapolation and step sizes \(\Delta x = 1\) and \(\Delta x = 1/2\). Explain why you obtain the exact result.