Practice — Numerical Analysis — FMN011 — 110519

The exam lasts 4 hours. A minimum of 35 points out of the total 70 are required to get a passing grade. These points will be added to those obtained in your two home assignments, and the final grade is based on your total score.

Justify all your answers and write down all important steps. Unsupported answers will be disregarded.

During the exam you are allowed a pocket calculator, but no textbook, lecture notes or any other electronic or written material.

1. **(4p)** If $x \in \mathbb{R}^n$ is an error vector, what vector norm should you use (1-norm, 2-norm, ∞-norm) if you want to construct an algorithm that requires the errors for each component to be less than a certain given number ϵ? Explain your choice.

2. **(5p)** Find the least squares line $y = mx + b$ for the following data:

x_k	0	1	2	3
y_k	-0.9	1.0	3.1	2.9

3. **(5p)** Select the most appropriate answer.

   (a) The error in a Lagrange interpolation polynomial depends mainly on

      i. value of f
      ii. spacing of data points
      iii. value of x_0

   (b) All previous computations can still be used when new data points are added in the following type of polynomial representations:

      i. Lagrange's
      ii. Bernstein's
      iii. Newton's

   (c) The development of a quadratic spline with n data points requires the use of

      i. $3(n-1)$ conditions
      ii. $3n$ conditions
      iii. $2n$ conditions

   (d) The power method can be considered

      i. a direct method
      ii. an iterative method
      iii. a root finding method

   (e) The Jacobi or Gauss-Seidel methods will not converge if

      i. A is not strictly diagonally dominant
      ii. the largest eigenvalue of the iteration matrix has absolute value greater than 1

iii. the initial guess is too far from the exact solution

4. It is known that the matrix

$$A = \begin{pmatrix} -8.4 & -12.7 \\ 4.1 & 9.6 \end{pmatrix}$$

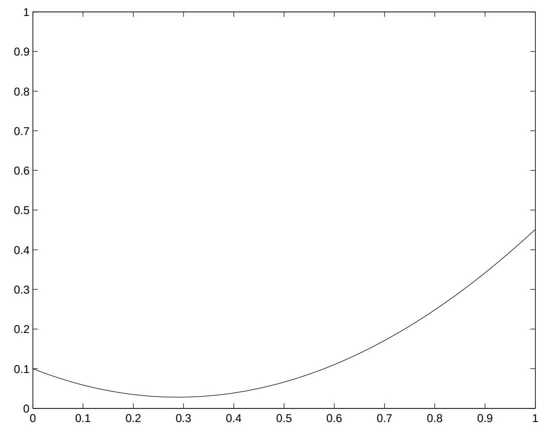
has an eigenvalue close to 6. We can apply the shifted inverse power method with an initial vector $(1, 0)$ to improve the accuracy of the eigenvalue.
a) **(3p)** Write down the formula for the iteration we must perform. (The direct power method is $x_{n+1} = Ax_n$.)
b) **(2p)** When we perform 4 steps we obtain the following results:

not normalized vector	normalized vector
(1.00 , 0.00)	(1.00 , 0.00)
(15.6 , -17.8)	(0.660 , -0.751)
(-31.2 , 35.3)	(-0.662 , 0.750)
(31.0 , -35.1)	(0.662 , -0.750)
(-31.0 , 35.1)	

Use these results to approximate the eigenvalue of A with 3 significant figures.

Note: The Rayleigh quotient is $\lambda = x^T Ax/(x^T x)$.

5. **(4p)** The function $g(t) = t^2 - e^{t/2} + 1.1$ is plotted in the interval $[0, 1]$.



Determine if the iteration $x^{(k+1)} = g(x^{(k)})$ will converge to a fixed point of g by checking if the function satisfies the conditions of the fixed-point iteration theorem.

6. **(5p)** If $\|\cdot\|$ is a matrix norm, then $\|A\| = 0 \Rightarrow A = 0$. Find a 2×2 nonzero matrix for which the spectral radius is zero, i.e., $\rho(A) = 0$. Is the spectral radius a matrix norm? Justify. (The spectral radius is $\rho(A) = \{\max_\lambda |\lambda|, Ax = \lambda x, x \neq 0\}$.)

7. **(5p)** Suppose A is a symmetric matrix, and consider the following MATLAB code:

```
function M = whatisit(A,k)
% A must be a square symmetric matrix
% k is the number of iterations
n = size(A,1);
R = A;
Q = eye(n);
for j = 1:k
   [Q,R] = qr(R*Q);
end
M = Q*R;
```

What will the (approximate) structure of matrix M be after a large number of iterations? To what values will its entries converge?

8. **(5p)** Suppose you want to compress an 8 by 1 vector by applying a DFT and deleting the higher frequencies so that you get a compression rate of 2:1. What entries of the transformed vector would you modify and how? You may illustrate your answer with a sketch or drawing. Justify your answer.

9. **(5p)** Fit the data

x	-1	0	1	2
y	1	0	0	-2

to the model $y = mx$ with least squares using SVD.

10. (a) **(2p)** The Shannon information formula is

$$I = -\sum_{i=1}^{k} p_i \log_2 p_i$$

Calculate the average least number of bits needed to code the matrix

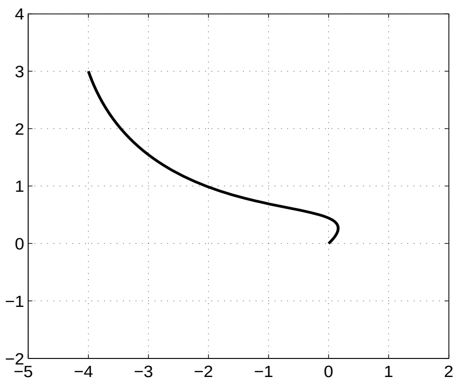
$$A = \begin{pmatrix} 4 & 7 & 7 \\ 9 & 0 & 8 \\ 8 & 8 & 7 \end{pmatrix}$$

(b) **(2p)** Construct the Huffman code for A.

(c) **(2p)** What is the average for this coding? What is the average if the standard binary system is used for the matrix entries?

11. The following plot shows a cubic Bézier curve.

a) **(2p)** Which of these parametric equations best represents the Bézier curve? Justify your choice.



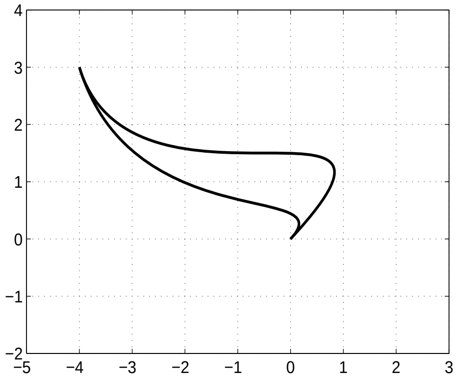
$$x = 3t(1-t)^2 - 9t^2(1-t) - 4t^3 \quad (1)$$
$$y = 3t(1-t)^2 + 3t^3$$

$$x = 3t(1-t)^2 + 9t^2(1-t) \quad (2)$$
$$y = 3t(1-t)^2 + 3t^3$$

$$x = 3t(1-t)^2 + 9t^2(1-t) - 4t^3 \quad (3)$$
$$y = -3t(1-t)^2 + 3t^3$$

b) **(2p)** Draw the convex hull of this curve.

c) **(2p)** Suggest a possible change of one control point (give old and new coordinates) so that the previous curve is modified to get the following plot (where we see both the old and the new curves).



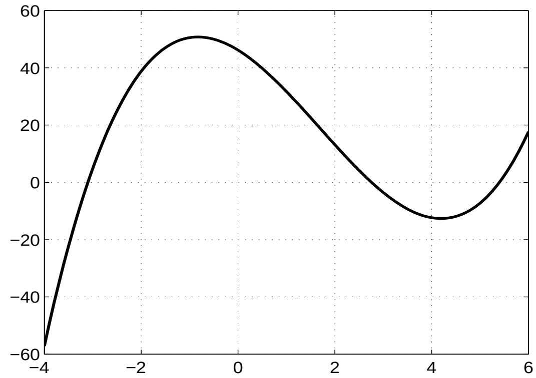
12. **(5p)** We wish to fit a piecewise quadratic interpolant

$$\begin{cases} q_1(x) = a_1x^2 + b_1x + c_1, & 0 \le x \le \pi/2 \\ q_2(x) = a_2x^2 + b_2x + c_2, & \pi/2 \le x \le \pi \end{cases}$$

to $y = \sin x$ at the nodes $\{0, \pi/2, \pi\}$. We desire to match the derivative of $y(x)$ at the center node. Write the system of 6 equations for the coefficients of the two interpolating polynomials. Do not solve the system.

13. **(5p)** Use a table of divided differences to find the interpolating polynomial for $f(x) = x^4 - 2x^3 + 2x - 1$ based on the nodes $\{0, 1, 2, 3\}$.

14. A plot of the function $f(x) = x^3 - 5.06x^2 - 10.392x + 46.2$ is



a) **(3p)** Use Newton's method, $x_{n+1} = x_n - f(x_n)/f'(x_n)$, to find the largest root of $f(x) = 0$ to within a relative error of no more than 10^{-3}, as estimated by

$$\rho_{n+1} = |x_{n+1} - x_n|/|x_{n+1}|.$$

b) **(2p)** Find the residual.