1. Explain why a divergent infinite series, such as
\[ \sum_{n=1}^{\infty} \frac{1}{n}, \]
can have a finite sum in floating-point arithmetic. At what point will the partial sums cease to change?

2. What condition ensures that the bisection method will find a zero of a continuous nonlinear function in the interval \([a, b]\)?

3. How many iterations are needed to guarantee an error no greater than \(10^{-9}\) if we start the bisection method in the interval \([2, 2.5]\)?

4. For double roots, the Newton-Raphson method converges linearly. What is the convergence rate for Newton-Raphson’s method for finding the root \(x = 2\) of each of the following equations?
   (a) \(f(x) = (x - 1)(x - 2)^2 = 0\)
   (b) \(f(x) = (x - 1)^2(x - 2) = 0\)

We can restore the quadratic rate for a root of multiplicity \(m\) by multiplying the function in Newton’s formula by \(m\).

5. List one advantage and one disadvantage of the bisection method compared with Newton’s method for solving a nonlinear equation in one dimension.

6. Write out Newton’s iteration for solving the equation \(x \sin x = 1\).

7. What methods does MATLAB’s \texttt{fzero}\ function use? And \texttt{fsolve}\ (in Python, \texttt{scipy.optimize.fsolve})?

8. Suppose you are using fixed-point iteration based on \(x = g(x)\) to find a solution \(x^*\) to a nonlinear equation \(f(x) = 0\). Which would be more favorable for the convergence rate: a horizontal tangent of \(g\) at \(x^*\) or a horizontal tangent of \(f\) at \(x^*\)?

9. Consider the function \(f(x) = x + \ln x\).
   (a) Plot the functions \(y = x\) and \(y = -\ln x\) to show \(f\) has a unique root \(P\) in \((0, \infty)\).
   (b) Show that if \(g(x) = -\ln x\) then \(|g'(P)| > 1\).
   (c) Can the root \(P\) be found using a fixed point iteration \(x = -\ln x\)?
(d) Can the root $P$ be found using a fixed point iteration $x = g(x)$, with a different $g$?

10. Make graphs showing all basic patterns of convergence and divergence of fixed-point iteration.

11. Plot the following functions in [-5,5] and discuss the convergence of the Newton-Raphson method to find their roots:
   
   (a) $f(x) = \arctan(x)$. (In Matlab, $\operatorname{atan}(x)$)
   
   (b) $f(x) = x^{1/3}$.
      (In Matlab, define the function as $\operatorname{sign}(x) \cdot \operatorname{abs}(x)^{(1/3)}$.)

12. Implement your own Newton-Raphson method and solve the system

   \[
   \begin{align*}
   u^2 + v^2 &= 1 \\
   (u - 1)^2 + v^2 &= 1
   \end{align*}
   \]

   Check that the convergence is quadratic.

13. Which fixed point iteration will converge faster to the root $x = 1$, starting from $x_0 = 0.5$?

   (a) $x_{k+1} = x_k - \frac{x_k - 1}{x_k^3 + 1}$
   
   (b) $x_{k+1} = x_k + \frac{x_k - 1}{5x_k^2 + 1}$

14. The polynomial

   \[ p(x) = x^4 - 6x^3 - 11x^2 + 2x - 28 \]

   has one positive root at $x = 7.497923191894927$. Use this fact to investigate if the method given by

   \[ x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})} \]

   has linear or quadratic convergence, or neither.

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