1. True or false:

(a) If \( A \) is nonsingular, then the number of solutions to \( Ax = b \) depends on the particular choice of vector \( b \).

(b) For a symmetric matrix \( S \), it is always the case that \( \|S\|_1 = \|S\|_\infty \).

(c) If a triangular matrix has a zero entry on its main diagonal, then the matrix is necessarily singular.

(d) The product of two upper triangular matrices is also upper triangular.

(e) If the rows of a square matrix are linearly dependent, then the columns of the matrix are also linearly dependent.

(f) If \( A \) is any \( n \times n \) matrix and \( P \) is any \( n \times n \) permutation matrix, then \( PA = AP \).

(g) For \( x \in \mathbb{R}^n \), \( \|x\|_1 \geq \|x\|_\infty \).

(h) If \( \det(A) = 0 \), then \( \|A\| = 0 \).

(i) The product of two symmetric matrices is also symmetric.

(j) \( \kappa_p(A) = \kappa_p(A^{-1}) \).

(k) Every nonsingular matrix \( A \) can be written as \( A = LU \), where \( L \) is lower triangular and \( U \) is upper triangular.

(l) A non-invertible matrix does not have an LU(P) factorization.

2. Given \( Ax = b \), what effect on the solution vector \( x \) results from

(a) Permuting the rows of \([A \ b]？

(b) Permuting the columns of \( A \)？

(c) Multiplying both sides of the equation from the left by a nonsingular matrix \( M \)？

3. Consider the matrix

\[
A = \begin{bmatrix}
4 & -8 & 1 & 2 \\
6 & 5 & 7 & 3 \\
0 & -10 & -3 & 5 \\
5 & -1 & 1 & 0
\end{bmatrix}
\]

What will the initial pivot in Gaussian elimination be if

(a) No pivoting is used?

(b) Pivoting is used?

4. Given \( n \times n \) matrices \( A \) and \( B \), what is the best way to compute \( A^{-1}B \)?
5. If $x$ is a column vector and $A$ is a matrix, which of the following computations require less work?

(a) $y = (xx^T)A$
(b) $y = x(x^TA)$

6. What is the inverse of a permutation matrix $P$?

7. Assume you have already computed the LU factorization, $PA = LU$. How would you use it to solve the system $A^Tx = b$?

8. Classify each matrix as well conditioned or ill conditioned:

(a) \[
\begin{pmatrix}
10^{10} & 0 \\
0 & 10^{-10}
\end{pmatrix}
\]
(b) \[
\begin{pmatrix}
10^{10} & 0 \\
0 & 10^{10}
\end{pmatrix}
\]
(c) \[
\begin{pmatrix}
10^{-10} & 0 \\
0 & 10^{-10}
\end{pmatrix}
\]
(d) \[
\begin{pmatrix}
1.0000001 & 2 \\
2 & 4
\end{pmatrix}
\]

9. In solving a linear system $Ax = b$, what is meant by the residual of an approximate solution $\hat{x}$? Does a small relative residual always imply that the solution is accurate?

10. Rank the following methods according to the amount of work required for solving most systems:

(a) Gaussian elimination
(b) $LU$ factorization followed by forward- and back-substitutions
(c) Explicit matrix inversion followed by matrix-vector multiplication

11. What quantity is minimized when using least squares to solve an overdetemined system $Ax \cong b$?

12. True or false:

(a) A linear least squares problem always has a solution.
(b) At the solution to a least squares problem $Ax \cong b$, the residual vector $r = b - Ax$ is orthogonal to the space generated by $A$ (i.e. $Ax$ for all $x$).

13. Let $A$ be an $m \times n$ matrix. Under what conditions on the matrix $A$ is the matrix $A^TA$ nonsingular?

14. In an overdetermined linear least squares problem $Ax \cong b$, where $A$ is an $m \times n$, if rank$(A) < n$, then which of the following situations are possible?
(a) There is no solution
(b) There is a unique solution
(c) There is a solution, but it is not unique

15. In solving an overdetermined least squares problem $Ax \approx b$, which would be a more serious difficulty: that the rows of $A$ are linearly dependent, or that the columns of $A$ are linearly dependent?

16. In fitting a straight line $y = x_0 + x_1 t$ to the three data points $(0, 0), (1, 0), (1, 1)$, is the least squares solution unique? Why?

17. What is the Euclidean norm of the minimum residual vector for the following linear least-squares problem?

$$
\begin{pmatrix}
1 & 1 \\
0 & 1 \\
0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\end{pmatrix}
\approx
\begin{pmatrix}
2 \\
1 \\
1 \\
\end{pmatrix}
$$

What is the solution vector for this problem?