ON SHAPE CONTROL OF CABLES UNDER VERTICAL STATIC LOADS

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Abstract

Under conditions where dynamic loads on, for example, suspension bridges and suspended electrical cables are negligible, it is sometimes reasonable to assume that the cables in such structures are under the action of vertical static loads. The static loads, which often constitute a predominant portion of the total loads, are mostly due to the self-weight of the cables and the structures they carry.

In order to facilitate the calculation of the shape of a cable, it is, in some situations, reasonable to assume that the cable can be divided into sections whose ends are rigidly supported at points with prescribed co-ordinates, and then determine the shape of such a section of the cable. There exist a number of useful classical solutions for the shape of a single cable that is rigidly supported at its ends, and under the action of downwardly directed vertical static loads such as gravity, distributed loads, point forces, or a combination of such loads.

Several different kinds of cables exist, and hence a constitutive model has to be established for each type of cable to govern its structural response to mechanical loads. In the present paper, we assume, in accordance with classical cable theory, that the analysed cables have zero flexural rigidity, and that the cables are either inextensible, or that they behave as a homogenous linearly elastic material in the axial direction.

In order for the solutions, presented in the present paper, to be as accurate as possible within the validity of the assumptions of classical cable theory, we let the solutions for the shape of linearly elastic cables, presented in Irvine and Sinclair [2] (and in Section 1.3 of Irvine [1]), constitute the foundation of the present work. It is then assumed that the external loads on the linearly elastic cables always include gravity, and in some cases also vertical point forces.

The assumption that a cable does not have any flexural rigidity, and that external loads are applied in the form of point loads, implies that the slope of the cable centerline is modelled as discontinuous at the points of application of the point loads. Although point loads do not exist in reality, and despite the fact that the flexural rigidity of a real cable may be locally significant in the vicinity of the region of contact between the cable and other structural elements, the classical solutions are nevertheless useful in many cases.

In applications such as suspension bridges and suspended electrical cables, it is usually desirable to control the shape of the loaded cables, in order to assure that one or more of the position co-ordinates of certain points of the cable centerline take prescribed values. The classical solutions for inextensible cables can, in some cases, be derived such that the shape of the cable can be controlled in accordance with the requirements. In many important applications, the assumption of inextensibility is, however, insufficient because the length of the cable increases significantly as a result of the internal stresses in the cable. Consequently, the shape of the cable may deviate significantly from that predicted by the model of the inextensible cable. The classical solutions of Irvine and Sinclair [2], offer limited possibilities to control the shape of the cable. This is due to the fact that most of the parameters that appear in the equations for the shape are expected to be known before hand. So, in order to control the shape, it is therefore necessary to determine the initially unknown parameters in accordance with the requirements, and it is the main purpose of the present paper to develop methods by which this can be done. We then assume that the shape of the cable is given by the solutions of Irvine and Sinclair [2], and the shape is controlled by calculating the initially unknown parameters in accordance with the requirements. An equation for every parameter to be determined is created, which result in a nonlinear system of equations that is solved numerically for the unknown parameters by using Newton's method. The starting values for

the numerical solution are usually provided from a solution of the corresponding problem involving an inextensible cable.

Special problems arise in those shape control problems that involve point forces as a result of the discontinuities of the slope of the cable centerline. However, it turns out that the shape of linearly elastic cables can be successfully controlled by the methods presented in the present paper, also in problems that involve point forces. Once the functions for the position co-ordinates of the cable centerline have been programmed on a computer, the equations to be solved in the shape control problems can often be programmed with only a few additional rows of program code.

Contents

Preface

The present paper is in partial fulfillment of the requirements for the Degree of Master of Science in Mechanical Engineering at Lund institute of Technology. The work was carried out at the department of Numerical Analysis in co-operation with the department of Mechanics at Lund Institute of Technology.

In technically important problems such as suspension bridges and suspended electrical cables, it is, in some situations, reasonable to assume that the cables involved are under the action of vertical static loads. There exist a number of useful classical solutions for the shape of cables under vertical static loads such as gravity, distributed loads, point forces, or a combination of such loads. It is often assumed that the cables do not have any flexural rigidity, and that they are either inextensible, or that they behaves as a homogenous linearly elastic material in the axial direction.

As presented in the references of the present paper, most of the parameters of the solutions for the shape of a linearly elastic cable are expected to be initially known. However, in many important problems, requirements are placed on the shape of the loaded cable. In such problems, the value of some of the parameters, such as the length of the unstrained cable, may nevertheless be initially unknown. The unknown parameters of the solutions for linearly elastic cables can, in many problems, be obtained approximately by calculating the shape of the cable according to an applicable theory of inextensible cables. However, no real cable is inextensible, which implies that the shape of the linearly elastic cable may deviate significantly from that predicted by the solution for the inextensible cable. It is therefore obvious that it would be convenient to have methods, by which the initially unknown parameters of the classical solutions for the shape of linearly elastic cables can be determined.

At the time of the beginning of the present work, I did not know of any such methods, and it seemed like an interesting technical and mathematical problem to undertake. It turned out that it is possible to derive methods by which the shape of linearly elastic cables can be accurately controlled in a straightforward way. Hopefully, the methods developed in the present paper can be of practical importance.

Acknowledgements

I wish to thank my supervisors, Claus Führer and Per Lidström, for their comments on the content and writing of the present paper.

1. Introduction

Load carrying cables are important structural and machine elements that are used in many applications such as suspension bridges, ski lifts, elevators, bicycle brake wires and fitness machinery. Another important application of cables is electrical cables, which are used in electrical power transmission lines. Although the primary task of such cables is to transfer electrical energy over long distances, these cables are loaded mechanically by, for instance, gravity and wind forces. Hence, electrical cables also have to be analysed mechanically in order to ensure that they can sustain the mechanical loads, and that they fulfil the requirements placed on the shape of the cables.

1.1 Cables under vertical static loads

Under conditions where dynamic loads on, for example, suspension bridges and suspended electrical cables are negligible, it is sometimes reasonable to assume that the cables in such structures are under the action of vertical static loads. The static loads, which often constitute a predominant portion of the total loads, are mostly due to the self-weight of the cables and the structures they carry. Problems that concern the determination of the shape of a cable that is under the action of vertical static loads represent an important class of cable problems.

In order to facilitate the calculation of the shape of a cable, it is, in some situations, reasonable to assume that cable can be divided into sections whose ends are rigidly supported at points with prescribed co-ordinates, and then determine the shape of such a section of the cable. There exist a number of useful classical solutions for the shape of a single cable that is rigidly supported at its ends, and under the action of downwardly directed vertical static loads such as gravity, distributed loads, point forces, or a combination of such loads.

The classical solutions for statically loaded cables, dealt with in the present paper, are such that calculation of the shape of the loaded cable is done without making any assumption regarding the shape of the unloaded cable. This is in contrast to the methods normally used to calculate the shape of loaded beams, shells and solids, since for such elements, it is usually necessary to make an assumption about their undeformed shape.

In general, the static equilibrium shape of a loaded cable differs significantly from that of a straight line, and the problem of determining the shape of a loaded cable is a geometrically nonlinear problem. An important feature of the classical solutions for cables is that the external loads are applied in full from the start, whereas for beams, shells and solids, the external loads are usually applied in steps when nonlinear problems are dealt with.

1.2 Shape control of cables under vertical static loads

In some applications, it is desirable to control the shape of a loaded cable in order to assure that one or more of the position co-ordinates, of certain points of the cable centerline, take prescribed values. As the first example of shape control of a cable, we take a cable that is loaded by gravity only. This cable is shown in Figure 1.1, and its shape has been controlled to fulfil the requirement that the distances d_h and d_v are to be equal to prescribed values.

As another example, we consider the main span section of the main cables of a suspension bridge (see Chapter 3 for a brief introduction to the basics of a suspension bridge).

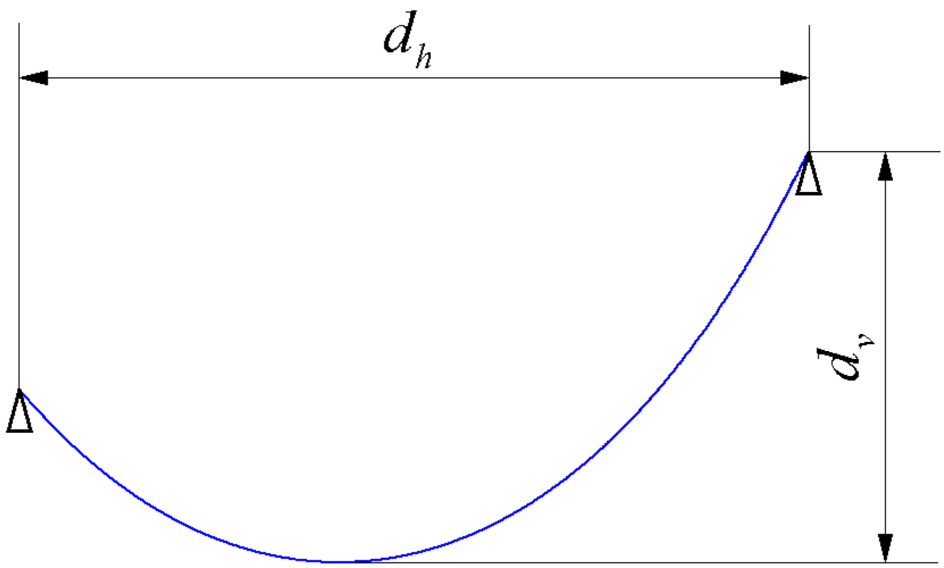


Figure 1.1: The cable is supported at its ends, and is under the action of gravity. This causes the cable to assume its static equilibrium shape, which is a function of the positions of the supports, the properties of the cable and the gravitational field. For every cable encountered in the present paper, it holds that the horizontal distance d_h, between the centers of the supports, is called the span of the cable, and the vertical distance d_v, between the highest and lowest points of the cable centerline, is called the sag of the cable.

The main cables carry the road deck and the vehicles on it via a finite number of vertical cables, so called hangers. The self-weight of the bridge deck and main cables constitute a large portion of the total load on the main cables. In order to determine the static equilibrium shape of the main cables at a certain temperature, and for the situation where there are no vehicles on the bridge, we assume that the main cables are loaded by gravity and a downwardly directed point force at the position of each hanger. Each point force represents the weight of the portion of the bridge deck that the pertinent hanger is assumed to carry, and, in some cases, also the self-weight of the hanger. It is necessary to be able to predict the horizontal location of each hanger in the strained bridge in order to predict the magnitude and location of each point force that is applied to the main cables. In addition, it is preferable if the horizontal location of each hanger centerline, as well as the distances d_h and d_v of the main cables, can be prescribed.

There exist some well known classical solutions for the shape of inextensible cables that are loaded by gravity only or by a downwardly directed distributed load along the span of the cable. As described in Chapters 2 and 3, the solutions for inextensible cables can, in some cases, be given on such form that the shape of the cable can be controlled in accordance with the requirements.

In many important applications, the assumption of inextensibility is insufficient because the length of the cable increases significantly as a result of the internal stresses in the cable. Consequently, the shape of the cable may deviate significantly from that predicted by the model of the inextensible cable. For linearly elastic cables, there exist classical solutions for single cables that are rigidly supported at their ends, and under the action of vertical static loads. In order for the solutions, presented in the present paper, to be as accurate as possible within the validity of the assumptions of classical cable theory, we let the solutions for the shape of linearly elastic cables, presented in Irvine and Sinclair [2] (and in Section 1.3 of Irvine [1]), constitute the foundation of the present work (see Chapters 4 and 6). It is then

assumed that the external loads on the linearly elastic cables always include gravity, and in some cases also vertical point forces. The classical solutions of Irvine and Sinclair [2], offer limited possibilities to control the shape of the cable. This is due to the fact that most of the parameters that appear in the equations for the shape are expected to be known before hand.

Chapters 5 and 7 are devoted to the main purpose of the present paper, which is to develop methods by which the shape of linearly elastic cables can be controlled. We then assume that the shape of the cable is given by the solutions of Irvine and Sinclair [2], and the shape is controlled by calculating the initially unknown parameters in accordance with the requirements. An equation for every parameter to be determined is created, which result in a nonlinear system of equations that is solved numerically for the unknown parameters by using Newton's method. Once a solution to a shape control problem has been obtained, we investigate the quality of the solution by comparing the results provided by the classical solution with the required result.

In many cases, a cable, whose shape has been controlled, is further analysed for other loads and conditions than those assumed for the shape control problem. It is then important to note that complex structures such as suspension bridges may have to be analysed as a whole system, and with special constitutive models for certain sections of the main cables. For instance, it may be necessary to use a different constitutive model for the portions of the main cables that are located on the tops of the towers of a suspension bridge.

1.3 The fundamental assumptions for the mechanical properties of the cables analysed in the present paper

Cables are usually characterised by having high axial tensile rigidity, and virtually no axial compressive rigidity, in the axial direction of the cable. In addition, it is often reasonable to assume that the flexural rigidity of a cable is negligible, and this assumption is also made for the cables analysed in the present paper.

Note, however, that the flexural rigidity of a cable may be important, and must then be taken into account. Without going into detail, it can be mentioned that, in some cases, the significance of the flexural rigidity of a cable depends, at least, on the cable tension and the radius of curvature of the cable centerline. If this is the case, then usually the flexural rigidity is significant if the cable tension is sufficiently low, or if the radius of curvature of the cable centerline is sufficiently small.

As stated above, we assume, in accordance with the classical solutions presented in Irvine and Sinclair [2], that the cables analysed in the present paper have zero flexural rigidity, and, in some problems, that the cables are under the action of vertical point forces. If it holds true that a cable has zero flexural rigidity, and that the cable is under the action of external point forces, then the slope of the cable centerline will be discontinuous at the points of application of the point forces. In reality, point loads do not exist, and it is often more reasonable to assume that the slope of the centerline of a cable is continuous at every point. In addition, it is possible that the flexural rigidity of a cable is locally significant in the vicinity of the regions of contact between the cable and other structural elements, where rapid changes of the radius of curvature of the cable centerline may occur.

Although there are some shortcomings of the classical cable theories, the assumptions of these theories are nevertheless reasonable in many problems, and in such problems, the classical solutions give results of good accuracy.

Most cables are not suited to carry substantial torsional moments, or undergo considerable torsional deformation. Therefore, cable structures are usually designed to avoid torsion of the

cables, although there are structures in which torsion of cables occur under certain conditions. The cables analysed in the present paper assume a shape that is located in a vertical plane, and it is expected that torsion of the cables does not occur.

A statically loaded cable with appreciable axial rigidity and no flexural and torsional rigidity is incapable of sustaining bending moments and torsional moments. Consequently, the cable can only carry the external loads by assuming a shape that causes the internal cross-sectional normal stress σ_N to be in equilibrium with all external loads and reaction forces that act on the cable. In the cable theories used in the present paper, it is assumed that the normal stress σ_N is evenly distributed over every cross-section perpendicular to the cable centerline. The normal stress $\sigma_N = f_{\sigma_N}(s)$ is a function of the arc-length co-ordinate s along the centerline of the unstrained cable, and the resultant of σ_N on a cross-section of the cable is given by

$$T = f_T(s) = \int_{A_0} f_{\sigma_N}(s)dA_0 = f_{\sigma_N}(s)A_0, \tag{1.1}$$

where A_0 is the constant cross-sectional area of the unstrained cable. It is assumed that the reduction, due to the action of σ_N, of the cross-sectional area of the cable can be neglected, which is why A_0 is used in Equation (1.1) instead of the cross-sectional area of the strained cable. The quantity $T = f_T(s)$ is the cable tension, which is the magnitude of the cross-sectional normal force

$$\boldsymbol{T} = T\boldsymbol{e_t} = f_T(s)\boldsymbol{f_{e_t}}(s), \tag{1.2}$$

where $\boldsymbol{e_t} = \boldsymbol{f_{e_t}}(s)$ is the tangent vector of the centerline of the loaded cable. We assume that $\boldsymbol{T}$ is located at the cable centerline. The force $\boldsymbol{T}$ is a tensile force, and it is defined that T is a positive scalar.

Several different kinds of cables exist, and hence an applicable constitutive model must be established for every type of cable to govern its structural response to mechanical loads. As described above, cables usually have high axial tensile rigidity and negligible flexural rigidity. In some situations, it may, therefore, be sufficient to assume that the analysed cable is inextensible, although no inextensible cables exist. However, there are many technically important applications in which the cable tension is sufficiently high to cause significant elongations of the cable. In such problems, the assumption of inextensibility is usually invalid, and thus it is necessary to use a constitutive model that considers the deformation of the cable. In many problems, it is reasonable to assume that the cables behave as a linearly elastic material in the axial direction of the cable.

The cables analysed in the present paper are either assumed to be inextensible or linearly elastic in the axial direction.

In summary, for every cable analysed in the present paper, we have assumed that:

- the cable has zero flexural rigidity
- the cross-sectional internal normal stress $\sigma_N = f_{\sigma_N}(s)$ is evenly distributed over every cross-section perpendicular to the cable centerline
- no torsion of the cable occurs since the strained cable is located in a vertical plane
- no bending moment or torsional moment exist in the cable

- the cable is either inextensible or behaves as a linearly elastic homogenous material in the axial direction

1.4 Outline of the present paper

Theoretical models for inextensible cables are, in some cases, well suited for shape control of cables. In Chapter 2, we derive the classical solution for a single cable under the action of gravity only. A large portion of the present paper deals with different sections of the main cables of suspension bridges. In Chapter 3, we describe a little bit about some of the basic principles of suspension bridges, and we derive the classical solution for an inextensible cable that is acted on by a constant distributed load that is assumed to represent the weight of the bridge deck.

There are problems where the assumption of inextensibility of cables is insufficient, and it is therefore necessary to include elasticity in the cable model. It is often sufficient to assume linear elasticity in many applications. In Chapter 4, we derive the solution for linearly elastic cables under pure gravity load, whereas in Chapter 6, the solution, in addition to gravity, is extended to include any number of vertical point forces.

As the models for linearly elastic cables, as given in the references, offer limited possibilities for shape control, we develop methods for controlling the shape of linearly elastic cables. Chapter 5 deals with shape control of cables under the action of gravity only, whereas Chapter 7 treats shape control of cables under gravity and vertical point forces.

In Chapter 8, we explain how the assumption of rigid supports, used in Chapters 2 to 7 of the present paper, may give accurate results, even in problems where the support flexibility may not be neglected.

The present paper also includes three appendices, of which Appendix A outlines the numerical solution method used for solving the nonlinear algebraic problems.

In the present paper, the problems given in the examples are solved by using the MATLAB-software. We give the MATLAB-code for selected functions and systems of equations in Appendix B.

Finally, Appendix C shows parts of the numerical results of some of the examples given in the present paper.

1.5 Remarks on the derivations of the classical solutions

The derivations, given in the present paper, of the classical solutions for the shape of a cable are in some respects different compared to those found in the references. As examples of this, we may mention that no dimensionless parameters are used in the present paper, and that the equations are derived in such a way that the location of the origin of the used co-ordinate system can be arbitrarily chosen. This is in contrast to how the equations are derived in the references, because in those, the location of the origin of the used co-ordinate system is usually the same in every problem.

2. Shape control of the inextensible catenary

A chain, or cable with no flexural rigidity, that is supported at its ends, and hanging under the action of a uniform gravitational field only, assumes a static equilibrium curve that is called a catenary. In this chapter, we derive the equation for the catenary assuming that the cable is inextensible, thus it is sometimes called the inextensible catenary. In reality, no cable is inextensible and, consequently, results obtained under the assumption of inextensibility are in some cases not very accurate. Certain models for inextensible cables are nevertheless useful in many situations because they are simple to use and, if their accuracy is not satisfactory, they are often sufficiently accurate to be used as providers of starting values for more sophisticated cable models. Another great advantage is that the equation for the inextensible catenary can be derived without involving the length of the cable. This approach is advantageous in problems where it is required that the span d_h and sag d_v of the cable have to assume prescribed values, since in such problems, the length of the cable is initially unknown. An equation for the length of the cable is then derived from the equation for the catenary. The so obtained equations for the catenary, and the length of the cable, may, for instance, be applicable in problems that involve chains or ropes that are to be hung between fence posts, or in problems concerning suspended electrical cables whose minimum and maximum allowable height above the ground is prescribed.

In the introduction to the inextensible catenary given in this chapter, we assume that the length of the cable is initially unknown. In problems where the length of the cable is initially known, we use in this paper the theory described in Chapter 4 in order to determine the shape of the cable. It is possible to derive an equation for the inextensible catenary assuming that the length of the cable is initially known, see for example Krenk [5], but this is not dealt with in the present paper. Descriptions of the inextensible catenary can, for example, be found in den Hartog [3], Meriam [4], Krenk [5] and Irvine [1], which are all used as references for the present discussion.

As shown in Figure 2.1 below, the ends of the cable are located at the fixed points A and B, respectively. We assume that the cable is of length L, and of constant self-weight per unit length $q_c = mg$, where m is the mass per unit length of the cable, and g is the gravitational acceleration. Along the cable, we have the arc-length co-ordinate $s,\ 0 \leq s \leq L$, for which we have chosen that $s = 0$ at point A, and $s = L$ at point B. The position co-ordinates of a point on the cable centerline are given by the Cartesian co-ordinates $x = f_x(s)$ and $z = f_z(s)$, relative to a co-ordinate system Oxz with horizontal x-axis and vertical z-axis (see Figure 2.1). In the present paper, the theory is derived such that the origin of the co-ordinate system Oxz can be chosen arbitrarily. The Cartesian co-ordinates of points A and B are, respectively, denoted (x_A, z_A) and (x_B, z_B).

We derive the equation for the inextensible catenary without involving the length of the cable. Consequently, the equation for z will be written as a function of x instead of s, which is possible since the function $x = f_x(s)$ is assumed to be bijective. This means that $s = f_s(x) = f_x^{-1}(x)$.

As seen from Figure 2.2, the assumption of inextensibility implies that the sine, cosine and tangent of the angle of inclination $\theta = f_\theta(s),\ -\pi/2 \leq \theta \leq \pi/2$, respectively, can be written as

$$\sin(\theta) = \frac{dz}{ds} \quad \text{(a)}, \qquad \cos(\theta) = \frac{dx}{ds} \quad \text{(b)}, \qquad \tan(\theta) = \frac{dz}{dx} \quad \text{(c)}. \tag{2.1}$$

Vertical equilibrium of the infinitesimal element shown in Figure 2.2 infers that

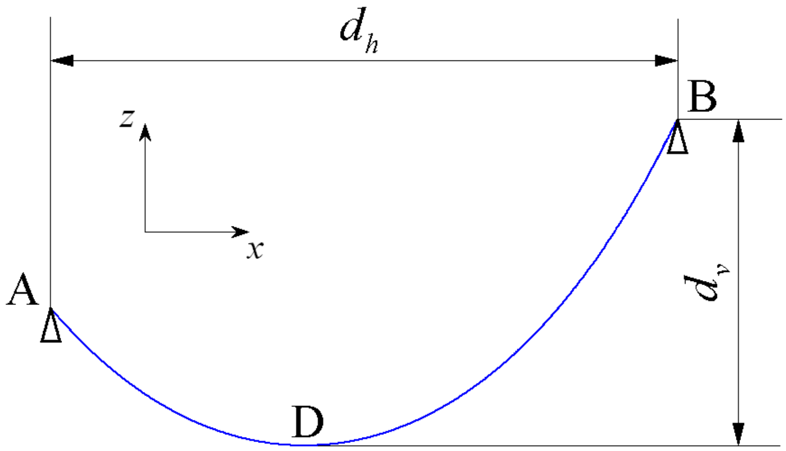


Figure 2.1: The end points of the cable centerline are called A and B, respectively, and the lowest point on the cable centerline, which is also a point with horizontal tangent vector, is called D. The span d_h and the sag d_v are shown for the cable which, in this figure, is of asymmetric shape since $z_A \neq z_B$. If, instead, the supports were on the same vertical level, that is if $z_A = z_B$, then the shape of the cable would be symmetric.

$$\frac{d}{ds}\left(T\frac{dz}{ds}\right) = mg, \tag{2.2}$$

and horizontal equilibrium requires that

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) = 0, \tag{2.3}$$

where $T = f_T(s)$ is the tension in the cable. Integration of Equation (2.3) with respect to s yields

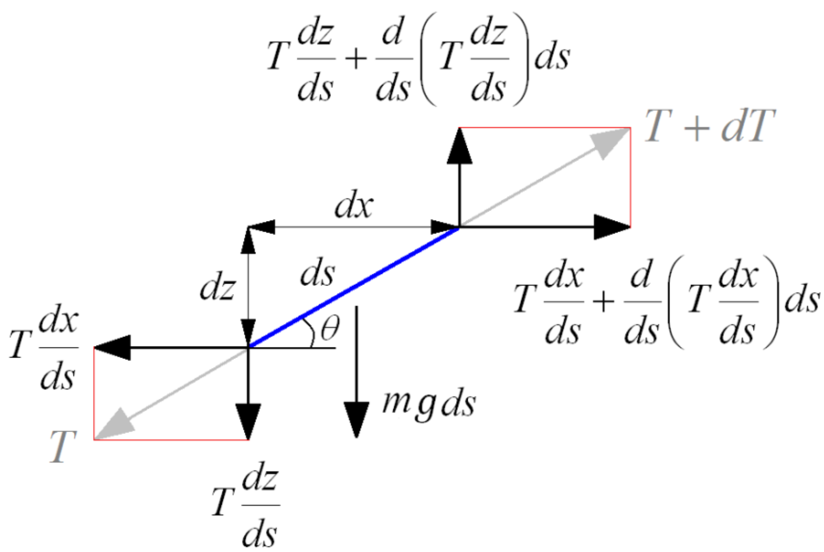


Figure 2.2: Equilibrium of an infinitesimal element of the inextensible cable under gravitational load.

$$T\frac{dx}{ds} = H, \tag{2.4}$$

where H is a constant of integration. Relation (2.1b) implies that H is the horizontal component of the cable tension T. The quantity H is constant, which is due to the fact that there are no external horizontal forces that act on the cable, other than the horizontal reaction forces at the ends of the cable. It holds that H is a positive number since it is assumed that $T > 0$, and because $-\pi/2 \leq \theta \leq \pi/2$, cf. relation (2.1b).

Insertion of $T = H\frac{ds}{dx}$ and $q_c = mg$ into Equation (2.2), and subsequent multiplication by ds/dx, gives

$$H\frac{d^2z}{dx^2} = q_c\frac{ds}{dx}. \tag{2.5}$$

By using the geometric relation $(ds)^2 = (dx)^2 + (dz)^2$, we can write Equation (2.5) as

$$H\frac{d^2z}{dx^2} = q_c\sqrt{1 + \left(\frac{dz}{dx}\right)^2}. \tag{2.6}$$

In order to simplify our notation, we introduce $a = dz/dx$, which yields

$$\frac{da}{dx} = \frac{q_c}{H}\sqrt{1 + a^2}. \tag{2.7}$$

After rewriting and then integrating Equation (2.7), we get

$$\int \frac{da}{\sqrt{1 + a^2}} = \int \frac{q_c}{H}dx \quad \Leftrightarrow \quad \ln(a + \sqrt{1 + a^2}) = \frac{q_c}{H}x + C_1, \tag{2.8}$$

where C_1 is a constant of integration. For later calculations, it is convenient to make the substitution $b = q_c x/H + C_1$. Then, by exponentiation of Equation (2.8), we obtain

$$a + \sqrt{1 + a^2} = e^b \quad \Leftrightarrow \quad \sqrt{1 + a^2} = e^b - a. \tag{2.9}$$

Squaring Equation (2.9), and subsequently solving for a, results in

$$a = \frac{e^b - e^{-b}}{2} = \sinh(b) \quad \Leftrightarrow \quad \frac{dz}{dx} = \sinh\left(\frac{q_c}{H}x + C_1\right). \tag{2.10}$$

By rewriting and then integrating Equation (2.10), we get

$$z = f_z(x) = \int \sinh\left(\frac{q_c}{H}x + C_1\right)dx = \frac{H}{q_c}\cosh\left(\frac{q_c}{H}x + C_1\right) + C_2, \tag{2.11}$$

where C_2 is another constant of integration. There are three unknown constants to be determined if q_c is initially known, namely C_1, C_2 and H. The constants C_1 and C_2 determine the location of the curve relative to the co-ordinate system. In the interval $x_A \leq x \leq x_B$, the

function $z = f_z(x)$ constitutes a mathematical model of the physical cable in static equilibrium. However, mathematically, the domain of definition of the function $z = f_z(x)$ is the whole set of real numbers and, in the entire domain of definition, there is one minimum point at which the derivative $dz/dx = 0.$ The minimum point of the function $z = f_z(x)$ is called $D,$ and it is assumed to be located at (x_D, z_D) (see Figure 2.1).

In many technically important problems, point D is on the physically relevant segment of the curve given by $z = f_z(x).$ In this case, D is the lowest point of the cable centerline, and $x_A \leq x_D \leq x_B$ (see Figure 2.1). However, in some cases, the minimum point of $z = f_z(x)$ is physically irrelevant.

It is often convenient to express the constants C_1 and C_2 in terms of x_D and $z_D,$ in such a way that the conditions $\frac{df_z}{dx}(x_D) = 0$ and $f_z(x_D) = z_D$ are fulfilled. To this end, we insert x_D and $dz/dx = 0$ into Equation (2.10), which is then solved to give $C_1 = -q_c x_D/H.$ Then, by inserting $x_D,$ z_D and C_1 into Equation (2.11), we get $C_2 = -H/q_c + z_D.$ The equations for dz/dx and z can now, in the physically relevant region, be written as

$$\frac{dz}{dx} = \frac{df_z}{dx}(x) = \sinh\left(\frac{q_c}{H}(x - x_D)\right), \; x_A \leq x \leq x_B, \tag{2.12}$$

$$z = f_z(x) = \frac{H}{q_c}\cosh\left(\frac{q_c}{H}(x - x_D)\right) - \frac{H}{q_c} + z_D, \; x_A \leq x \leq x_B. \tag{2.13}$$

It is necessary to determine the co-ordinates x_D and $z_D,$ as well as the horizontal component H of the cable tension, before the solution is complete. The method described here for determining these constants concerns problems where the span d_h and sag d_v of the cable are prescribed, as shown in Figure 2.1. We therefore assume that the co-ordinates $x_A,$ $z_A,$ z_B and z_D are prescribed, and that $x_B = x_A + d_h,$ whereas H and x_D have to be calculated.

In some cases, we can easily calculate $x_D.$ If, for example, the minimum point D is located at end point A (see Figure 2.3 on page 12), then, by definition, A is the same point as D and, consequently, $x_D = x_A.$ For a cable of symmetric shape, the x co-ordinate of point D can be calculated as $x_D = (x_A + x_B)/2.$ With x_D known, we determine H by inserting $x_D,$ z_D and the co-ordinates of one of the end points of the cable, into Equation (2.13), which gives a scalar nonlinear equation that is solved numerically for $H.$

If, on the other hand, the shape of the cable is asymmetric with neither point A nor point B being a minimum point, then we do not initially know the value of the co-ordinate x_D of point D and, therefore, both x_D and H must be calculated. In the present paper, this is done by inserting the co-ordinates of each end point into Equation (2.13), thus creating a system of two nonlinear equations which is solved numerically for x_D and $H.$ These equations are given by

$$0 = -z_A + \frac{H}{q_c}\cosh\left(\frac{q_c}{H}\left(x_A - x_D\right)\right) - \frac{H}{q_c} + z_D \tag{2.14a}$$

$$0 = -z_B + \frac{H}{q_c}\cosh\left(\frac{q_c}{H}\left(x_B - x_D\right)\right) - \frac{H}{q_c} + z_D. \tag{2.14b}$$

As illustrated in Example 2.2 below, the starting values needed for the numerical solution, H_0 and $x_{D,0},$ can be obtained by plotting, in the same figure, the plane given by the function

$$f_0(H, x_D) = 0,$$

together with the function graphs of the two functions

$$f_1(H, x_D) = -z_A + \frac{H}{q_c} \cosh\left(\frac{q_c}{H}\left(x_A - x_D\right)\right) - \frac{H}{q_c} + z_D \tag{2.15a}$$

$$f_2(H, x_D) = -z_B + \frac{H}{q_c} \cosh\left(\frac{q_c}{H}\left(x_B - x_D\right)\right) - \frac{H}{q_c} + z_D. \tag{2.15b}$$

Useful values of H_0 and $x_{D,0}$ exist in the vicinity of the physically relevant point of intersection of the graphs of the functions f_0, f_1 and f_2. It is then noted that H is a positive number.

In some situations, it might be more convenient to solve for the quotient H/q_c instead of solving for H. It is then easier to calculate H for various values of q_c if several different kinds of cables or chains are to be evaluated in an application where certain geometric parameters may not be changed.

With the aid of Equation (2.12), and introducing the integration variable $x^* = x$, the length of a section of the cable, as measured by the arc-length $s = f_s(x)$, can now be calculated as

$$s = f_s(x) = \int_{x_A}^{x} \sqrt{1 + \left(\frac{dz}{dx^*}\right)^2} dx^* = \int_{x_A}^{x} \sqrt{1 + \left(\sinh\left(\frac{q_c}{H}(x^* - x_D)\right)\right)^2} dx^* =$$

$$= \left[\frac{H}{q_c} \sinh\left(\frac{q_c}{H}(x^* - x_D)\right)\right]_{x_A}^{x} =$$

$$= \frac{H}{q_c}\left(\sinh\left(\frac{q_c}{H}(x - x_D)\right) - \sinh\left(\frac{q_c}{H}(x_A - x_D)\right)\right). \tag{2.16}$$

Consequently, the total length of the cable is given by

$$L = f_s(x_B) = \frac{H}{q_c}\left(\sinh\left(\frac{q_c}{H}(x_B - x_D)\right) - \sinh\left(\frac{q_c}{H}(x_A - x_D)\right)\right). \tag{2.17}$$

Since, at this stage, the function $s = f_s(x)$ is known, we can calculate the cable tension according to

$$T = f_T(x) = H\frac{ds}{dx} = H \cosh\left(\frac{q_c}{H}(x - x_D)\right). \tag{2.18}$$

By inserting Equation (2.18) into Equation (2.13), we get an alternative expression for the cable tension, which is expressed as

$$T = H + q_c(z - z_D). \tag{2.19}$$

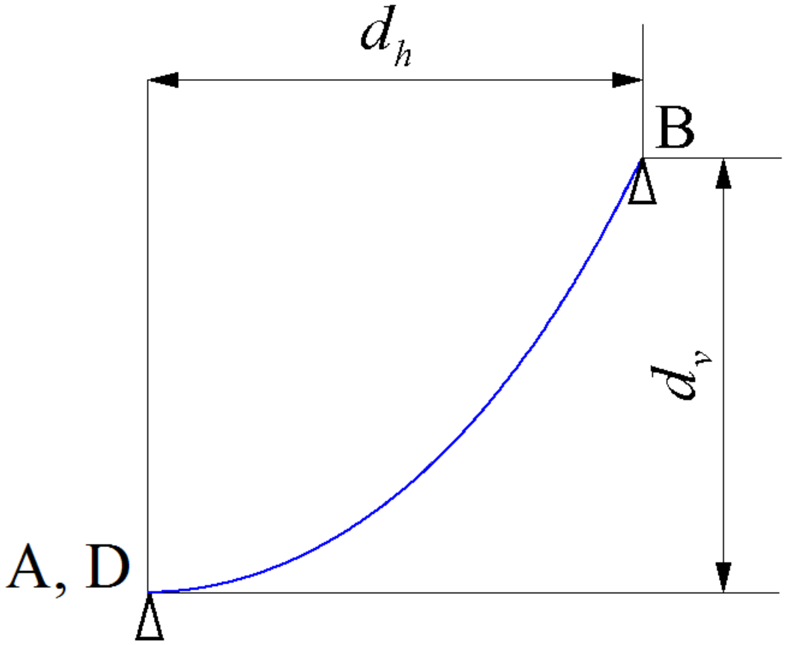


Figure 2.3: The tangent vector of the cable centerline is horizontal at A. Since this cable is inextensible, it holds that the shape of the cable is the same as that of the right half of an inextensible cable of symmetric shape with the sag $d_v^{\mathrm{sym}} = d_v$, and the span $d_h^{\mathrm{sym}} = 2d_h$. Point D is defined as the point on the cable centerline where the tangent vector is horizontal. This infers that A is the same point as D in this case.

Example 2.1, I

A cable of symmetric shape, and self-weight per unit length $q_c = 40$ N/m, has a span of $d_h = 100$ m, and a sag of $d_v = 2$ m. The cable is supported at its ends, and it hangs under the action of gravity only. Assuming that the cable is inextensible, calculate H and the length L of the cable.

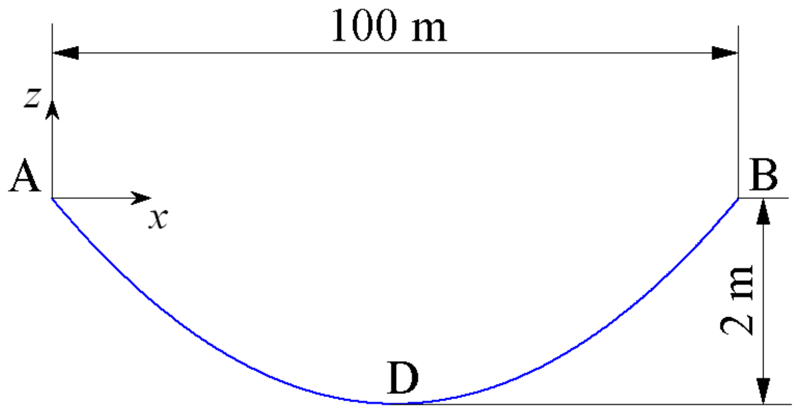


Figure E2.1.1: The shape of the cable, as illustrated prior to the calculation of H.

Solution

According to the theory of the inextensible catenary, as described in the present chapter, the location of the origin of the co-ordinate system may be chosen arbitrarily. In this example, we choose to locate the origin at point A. Consequently, the co-ordinates of the end points of the cable are

$$(x_A, z_A) = (0, 0) \text{ m} \quad \text{and} \quad (x_B, z_B) = (100, 0) \text{ m},$$

respectively. The fact that the cable is of symmetric shape infers that the lowest point of the cable centerline is located at

$$(x_D, z_D) = \left(\frac{x_A + x_B}{2}, -d_v\right) = (50, -2) \text{ m}.$$

In order to determine H, we insert x_B, z_B, x_D and z_D into Equation (2.13), which gives

$$0 = -z_B + \frac{H}{q_c} \cosh\left(\frac{q_c}{H}(x_B - x_D)\right) - \frac{H}{q_c} + z_D. \tag{E2.1.1}$$

Equation (E2.1.1) is then solved numerically for H to give

$$H = 25013 \text{ N}.$$

The starting value $H_0 = 25000$ N, needed for the numerical solution, is obtained graphically.

Now that H has been determined, the length of the cable can be calculated according to Equation (2.17), which yields

$$L = 100.107 \text{ m.}$$

The values of H and L, stated above, were calculated according to a theory which assumes that the cable is inextensible. Since the cable is of considerable length and self weight, and because the shape of the cable is shallow, the sag may be noticeably deeper than 2 meters if the cable is axially elastic, and the length of the unstrained cable is taken to be $L = 100.107$ m. The theory presented in Chapter 4 includes elastic deformation of the cable, and this theory is utilised when we continue the present example in order to calculate the deformations of the cable.

The technical and geometrical data of the cable in the present example are the same as those of Example 2.3 in Irvine [1]. In that example, the additional horizontal component of the cable tension, and the additional deflection, caused by a downwardly directed point force of magnitude $F_1 = 20$ kN, acting at midspan of an axially elastic cable, are calculated according to the theory presented in Chapter 2 of Irvine [1]. The length of the unstrained cable is not calculated or otherwise used in that example. In reality, it may be important to be able to calculate the length of an unstrained elastic cable, and methods for doing that are presented in Chapters 5 and 7 of the present paper.

Example 2.2, I

An inextensible cable of asymmetric shape, and self-weight per unit length $q_c = 40$ N/m, is supported at its ends, and hanging under the action of gravity only. The span of the cable is $d_h = 100$ m, and the sag is $d_v = 4$ m. As shown in Figure E2.2.1, point A is located 2 meters above point D, whereas point B is 4 meters above point D.

Calculate the horizontal component of the cable tension H, the co-ordinate x_D and the length L of the cable. Plot $x = f_x(s)$ versus $T = f_T(s)$.

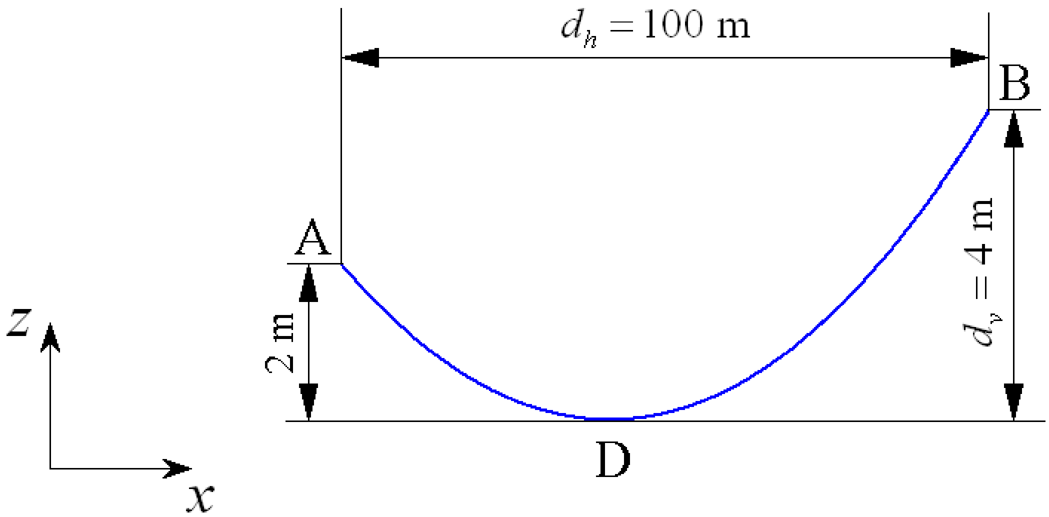


Figure E2.2.1: The shape of the cable of asymmetric shape, as illustrated prior to the calculation of the initially unknown parameters.

Solution

In this example, we chose to locate the co-ordinate system so that the co-ordinates of the end points of the cable, respectively, are

$$(x_A, z_A) = (125, 26) \text{ m} \quad \text{and} \quad (x_B, z_B) = (225, 28) \text{ m}.$$

Hence, the z co-ordinate of the lowest point of the cable centerline is

$$z_D = z_A - 2 = 24 \text{ m}.$$

We calculate H and x_D by solving the system of equations (2.14). The starting values for the numerical solution, H_0 and $x_{D,0}$, are obtained graphically by plotting the graphs of the functions f_1 and f_2, defined by Equations (2.15), together with the plane given by the function $f_0(H, x_D) = 0$, as shown in Figures E2.2.4a and E2.2.4b. From Figure E2.2.4b we get $H_0 = 1.7 \cdot 10^4$ N and $x_{D,0} = 165$ m. Solving the system of equations (2.14) for H and x_D yields

$$H = 1.7178 \cdot 10^4 \text{ N} \quad \text{and} \quad x_D = 166.431 \text{ m}.$$

Figure E2.2.2 shows the shape of the cable, whereas the convergence behaviour of the numerical calculation of H and x_D is shown in Figure E2.2.5.

Since H is now determined, the length of the cable is calculated according to Equation (2.17), which gives

$$L = 100.246 \text{ m}.$$

The cable tension is plotted in Figure E2.2.3.

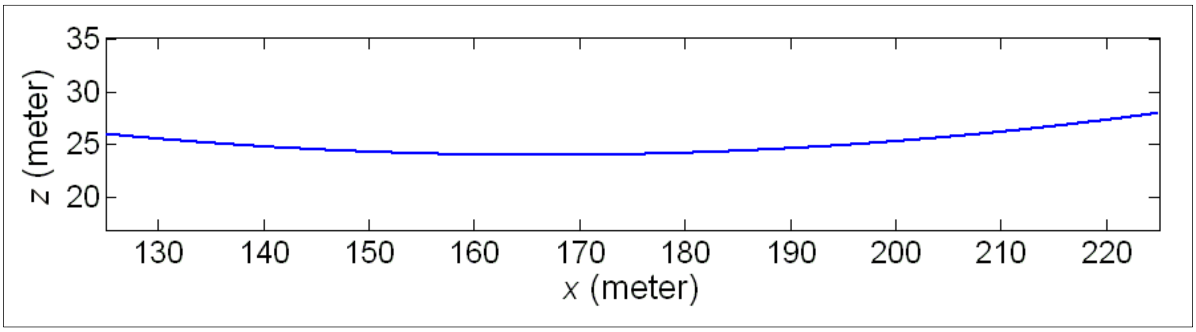


Figure E2.2.2: The shape of the inextensible cable.

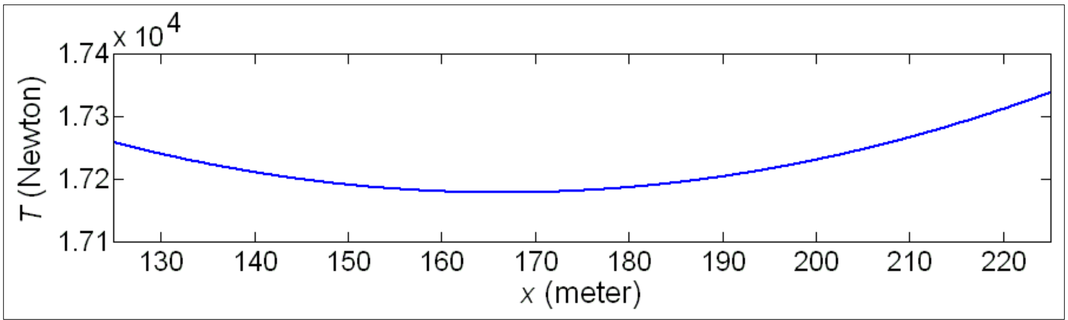


Figure E2.2.3: The cable tension $T = f_T(x)$ versus the co-ordinate x.

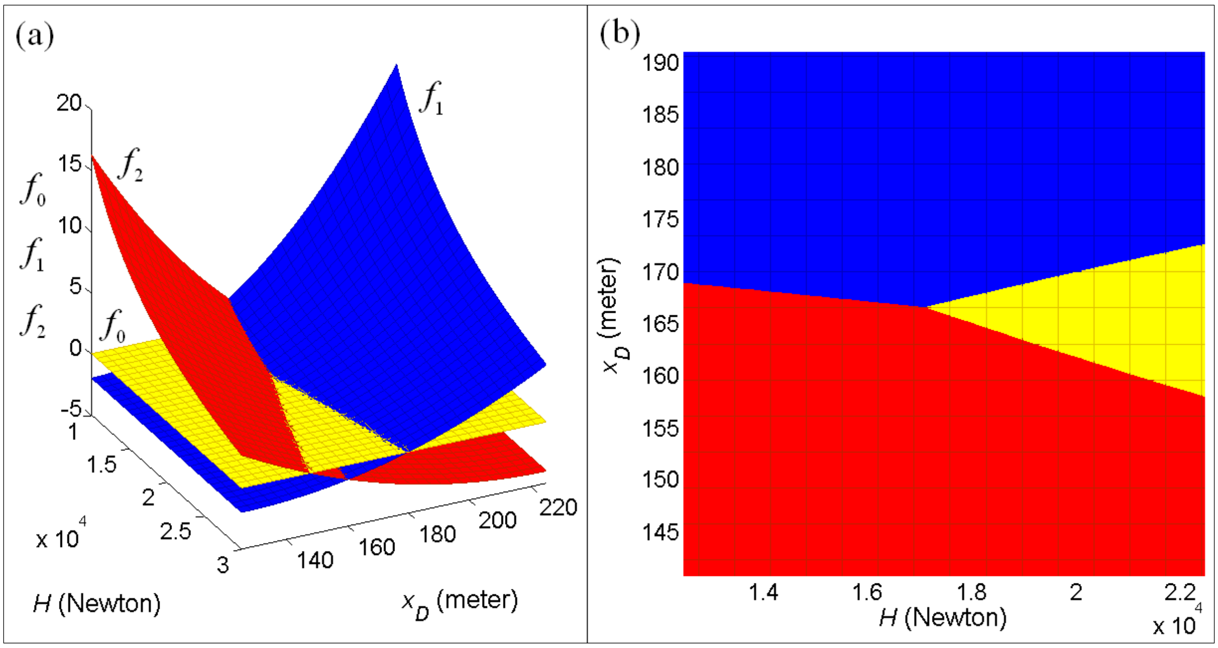


Figure E2.2.4a: The starting values for the numerical calculation of H and x_D can be obtained graphically by plotting, in the same figure, the plane $f_0(H, x_D) = 0$ and the graphs of the functions $f_1(H, x_D)$ and $f_2(H, x_D)$, defined by Equations (2.15).
Figure E2.2.4b: We get the starting values $H_0 = 1.7 \cdot 10^4$ and $x_{D,0} = 165$, at the point of intersection of the function graphs.

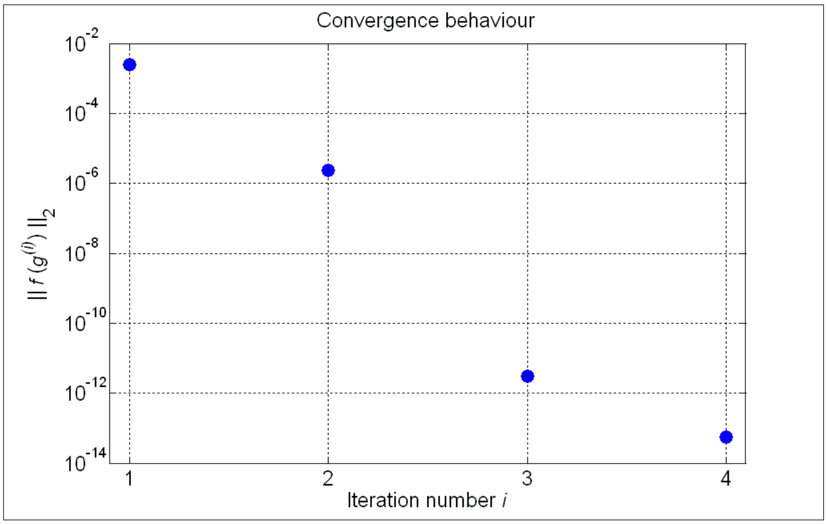


Figure E2.2.5: Convergence behaviour of the numerical calculation of H and x_D.

3. Inextensible cable under distributed load

Cables that support vertical external loads can, for example, be found in suspension bridges in which the weight of the road deck, plus the weight of the vehicles on it, is transferred to the main cables via a finite number of vertical cables, so called hangers. The structural analysis of cables is, in some cases, greatly simplified if the vertical external load can be approximated by a distributed vertical load that is given by a simple mathematical function (see Figure 3.1).

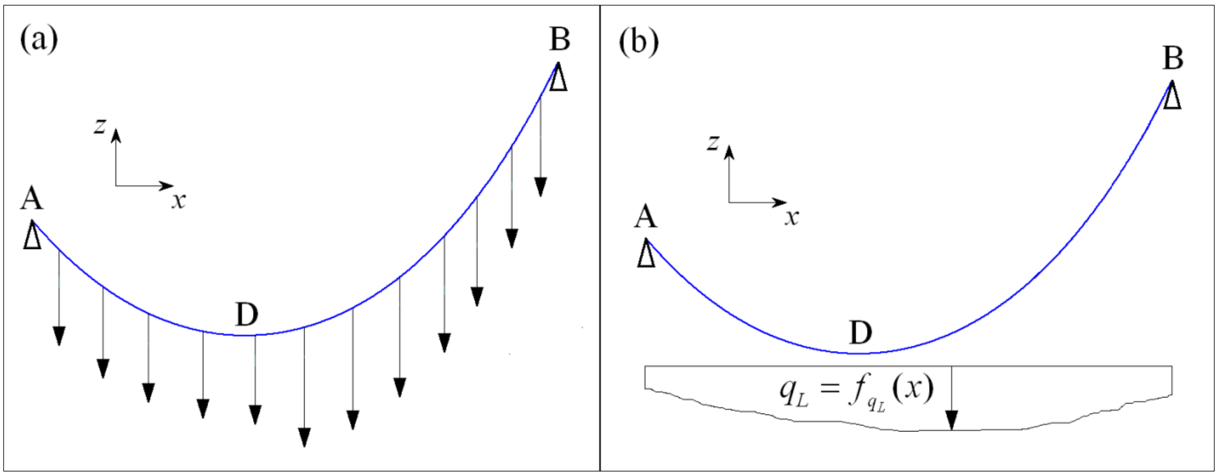


Figure 3.1: Determination of the static equilibrium shape of a cable can, in some cases, be simplified if the downwardly directed external loads, which are approximated as point forces in Fig. 3.1a, can be further approximated by a continuously distributed load, as shown in Fig. 3.1b.

Although it is possible to include axial elasticity in the analysis of cables which are under the action of distributed loads, we derive, for simplicity, in the present chapter an equation for the shape of the cable assuming that the cable is inextensible. The fact that the influence of elastic strains is not included in the analysis, often infers that the solution for the shape of the cable is not very accurate. However, in some problems, this equation is still very useful since it is sufficiently accurate to be used as a provider of starting values for more sophisticated theories of elastic cables. The equation also provides an easy means for quickly getting a rough estimate of the static equilibrium shape of a cable under the action of a vertical distributed load.

We now assume that the cable element shown in Figure 3.2 below, in addition to gravity, is also loaded by the distributed vertical external load $q_L = f_{q_L}(x)$. Vertical equilibrium of the element depicted in Figure 3.2 can be written as

$$\frac{d}{ds}\left(T\frac{dz}{ds}\right)ds = q_c ds + q_L dx \quad \Leftrightarrow \quad \frac{d}{ds}\left(T\frac{dz}{ds}\right) = q_c + q_L\frac{dx}{ds}. \tag{3.1}$$

Equation (3.1) is now multiplied by ds/dx, and T is replaced by $T = H ds/dx$ according to Equation (2.4), which result in

$$H\frac{d^2z}{dx^2} = q_c\frac{ds}{dx} + q_L. \tag{3.2}$$

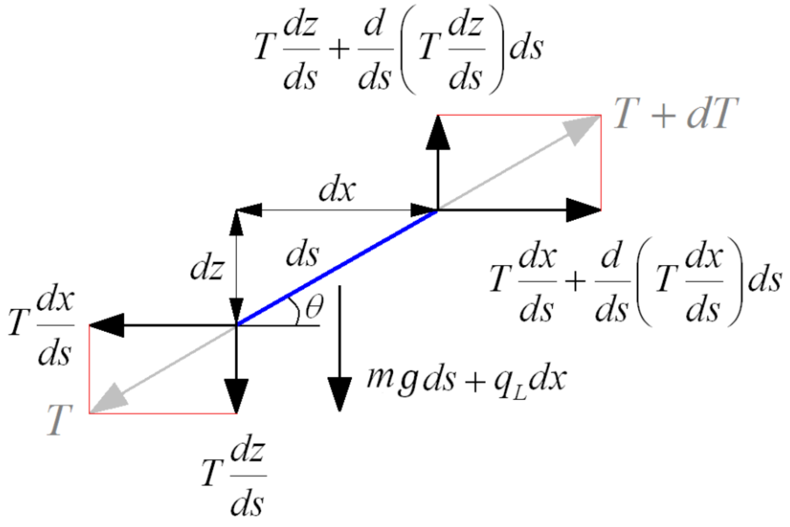


Figure 3.2: Equilibrium of an infinitesimal element of the inextensible cable under the action of gravity and the external load $q_L = f_{q_L}(x)$.

Making use of the geometric relation $(ds)^2 = (dx)^2 + (dz)^2$ in Equation (3.2), yields

$$H\frac{d^2z}{dx^2} = q_c\sqrt{1+\left(\frac{dz}{dx}\right)^2} + f_{q_L}(x). \tag{3.3}$$

It is in most cases not possible to find an analytic solution to Equation (3.3), which therefore may have to be solved numerically. However, in many applications, the distributed load q_L is significantly larger than the self weight of the cable q_c and, in addition, the shape of the cable is usually required to be quite shallow. Therefore, the self weight of the cable is usually either neglected or included approximately as a constant distributed external load, which is added to the primary external distributed load (see Example 3.1). Thus, by setting $q_c = 0$ in Equation (3.3), we get

$$H\frac{d^2z}{dx^2} = f_{q_L}(x). \tag{3.4}$$

The function $z = f_z(x)$ is obtained by integrating Equation (3.4) twice with respect to x.

Next, we will show how the function $z = f_z(x)$ can be obtained for a technically important load distribution $q_L = f_{q_L}(x)$.

3.1 Constant load distribution

The distributed weight of the bridge deck in a suspension bridge is, for the situation where no vehicles are on the bridge, often assumed to be given by a constant function of x. For this load, we set $q_L = f_{q_L}(x) =$ *constant*.

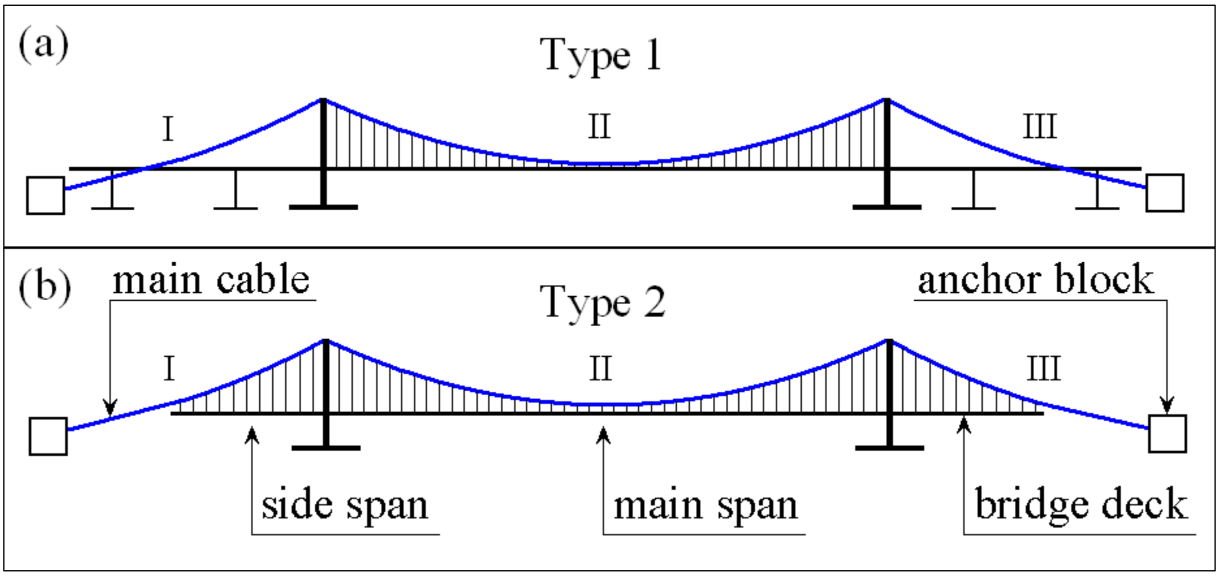


Figure 3.3: Two possible types of symmetric suspension bridges. (a) The side span section of the bridge deck rest on columns. This means that the side span sections of the main cables do not carry any portion of the bridge deck. (b) The side span sections of the bridge deck are carried by the main cables.

Figure 3.3 shows some basic parts of two types of conventional suspension bridges. Both bridges shown in the figure are symmetric since the towers, and the side spans, of each bridge are of equal height and length, respectively. If instead, for example, the side span sections of the bridge deck are of different length, or if the towers are of different height, then the bridge is asymmetric.

On either side of each bridge shown in Figure 3.3, the bridge deck is carried by a main cable via vertical hangers. It is assumed that the main cables cannot move relative to the towers of the bridge. Therefore, the main cables are thought of as being divided into three sections, here denoted I, II and III, which are analysed separately. Section II is the main span section of the main cables, whereas Sections I and II are the respective side span sections of the main cables.

In the present paper, we assume that both main cables of a suspension bridge are identical with respect to geometry, material properties and loading. Therefore, we only analyse one of the main cables in the examples that concern suspension bridges.

Most existing suspension bridges are symmetric, or nearly symmetric. If they are not symmetric, it is then usually the side spans that are of different length. However, in Example 3.3 of the present paper, we analyse Section II of the main cables of a suspension bridge that has towers of different height. This example is, therefore, mostly of theoretical interest.

For $q_L =$ *constant*, Equation (3.4) is integrated twice with respect to x, which yields

$$\frac{dz}{dx} = \frac{df_z}{dx}(x) = \frac{1}{H}\int q_L dx = \frac{q_L x}{H} + C_1, \tag{3.5}$$

$$z = f_z(x) = \int \frac{q_L x}{H} dx + \int C_1 dx = \frac{q_L x^2}{2H} + C_1 x + C_2, \tag{3.6}$$

where C_1 and C_2 are constants of integration. These constants determine how the graph of the function $z = f_z(x)$ is located relative to the co-ordinate system. As in Chapter 2, the

minimum point of the function $z = f_z(x)$ is called D, and it is assumed to be located at (x_D, z_D). In order to determine the constants C_1 and C_2 in terms of x_D and z_D, such that the conditions $\frac{df_z}{dx}(x_D) = 0$ and $f_z(x_D) = z_D$ are fulfilled, we insert x_D and $\frac{df_z}{dx} = 0$ into Equation (3.5), which is then solved to give

$$C_1 = -\frac{q_L}{H}x_D. \tag{3.7}$$

Then, by inserting x_D, z_D and C_1 into Equation (3.6), we get

$$C_2 = \frac{q_L}{2H}x_D^2 + z_D. \tag{3.8}$$

The equations for $\frac{df_z}{dx}(x)$ and $f_z(x)$ can now, in the physically relevant region, be written as

$$\frac{dz}{dx} = \frac{df_z}{dx}(x) = \frac{q_L}{H}(x - x_D),\ x_A \leq x \leq x_B, \tag{3.9}$$

$$z = f_z(x) = \frac{q_L}{2H}(x - x_D)^2 + z_D,\ x_A \leq x \leq x_B. \tag{3.10}$$

It is seen that with a constant load distribution q_L, the shape of the cable is given by a parabolic function of x, which is a classic result. The horizontal component H of the cable tension, as well as the co-ordinate x_D, must be determined before the solution is complete. Assuming that x_A, z_A, x_B, z_B and z_D are prescribed, we must calculate H and x_D to complete the solution. For a cable of symmetric shape, or a cable whose lowest point is an end point at which the derivative $dz/dx = 0$, this can be done by inserting the prescribed co-ordinates of one of the end points of the cable, and the known, or easily calculated, value of x_D into Equation (3.10), which leads to a single variable linear equation that is solved for H.

If, instead, the shape of the cable is asymmetric, we do not initially know the value of x_D. In a situation where the co-ordinates of both end points of the cable centerline, and the z co-ordinate of the lowest point of the cable centerline are prescribed, we can determine H and x_D by inserting z_D and the co-ordinates of each end point into Equation (3.10). Hence, we get a system of two nonlinear equations that is to be solved for x_B and H. These equations can be written as

$$0 = -z_A + \frac{q_L}{2H}(x_A - x_D)^2 + z_D \tag{3.11a}$$

$$0 = -z_B + \frac{q_L}{2H}(x_B - x_D)^2 + z_D. \tag{3.11b}$$

As illustrated in Example 3.2, there are problems in which H is prescribed instead of z_D. In such problems, it is usually convenient to determine the constants C_1 and C_2 by inserting the prescribed values of x_A, z_A, x_B, z_B and H into Equation (3.6), which results in the following system of linear equations

$$x_A C_1 + C_2 = z_A - \left(\frac{q_L}{2H}\right) x_A^2 \tag{3.12a}$$

$$x_B C_1 + C_2 = z_B - \left(\frac{q_L}{2H}\right) x_B^2. \tag{3.12b}$$

Once Equations (3.12) have been solved for C_1 and C_2, x_D is obtained from Equation (3.7) as

$$x_D = -\frac{C_1 H}{q_L}. \tag{3.13}$$

Our next objective is to find the expression for the cable tension $T = f_T(x)$. We see in Figure 3.4 that $T = \sqrt{H^2 + V^2}$, and that $V = H\tan(\theta)$. With the aid of Equations (2.1c) and (3.9), it is concluded that

$$\tan(\theta) = \frac{dz}{dx} = \frac{q_L}{H}(x - x_D). \tag{3.14}$$

The expression for the cable tension can now be written as

$$T = f_T(x) = \sqrt{H^2 + q_L^2 (x - x_D)^2}. \tag{3.15}$$

The length of a section of the cable, as measured by the arc-length $s = f_s(x)$, is calculated according to

$$s = f_s(x) = \int_{x_A}^{x} \sqrt{1 + \left(\frac{dz}{dx^*}\right)^2} dx^* = \int_{x_A}^{x} \sqrt{1 + \left(\frac{q_L}{H}(x^* - x_D)\right)^2} dx^*, \tag{3.16}$$

where $x^* = x$ is an integration variable. In the present paper, we calculate the integral of Equation (3.16) numerically, although it is possible to obtain an analytical expression for this integral.

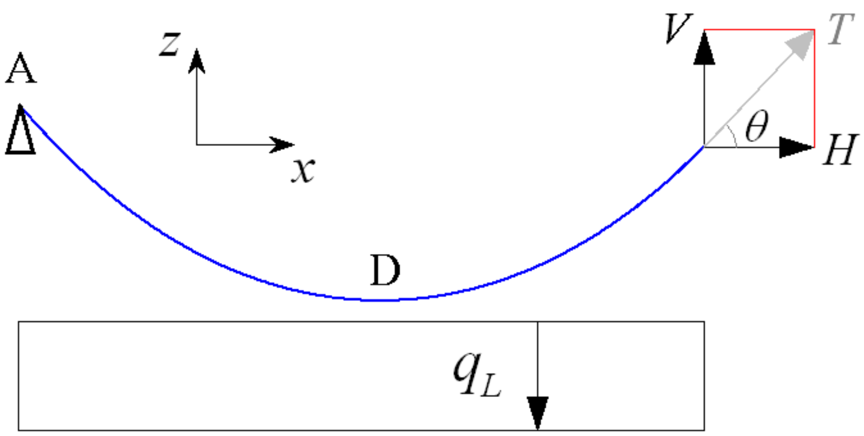


Figure 3.4: A portion of a cable under the action of the constant distributed load q_L is studied. The section force components $\boldsymbol{H}$ and $\boldsymbol{V}$, which act on the end section of this portion of the cable, are shown together with their resultant $\boldsymbol{T}$ of magnitude $T = \sqrt{H^2 + V^2}$.

Example 3.1, I

The main span section of the main cables of a large suspension bridge is of symmetric shape with a span $d_h = 1300$ m, and a sag $d_v = 150$ m, when no vehicles are on the bridge.

It is assumed that the towers are rigid and that the main cables cannot move relative to the towers. We will therefore analyse certain sections of the main cables separately. The subsequent analysis is performed per main cable, and it concerns the main span section of the main cables.

The road deck is horizontal, and its constant weight per unit length along the main span is $q_{L,\text{total}} = 2.2 \cdot 10^5$ N/m. Each main cable is assumed to carry half the weight of the portion of the road deck that stretches horizontally between x_A and x_B. Consequently, we have that

$$q_L = \frac{q_{L,\text{total}}}{2} = 1.1 \cdot 10^5 \text{ N/m}.$$

(a) Neglect the weight of the main cables and hangers. Calculate H, the maximum cable tension $T_{\max}$ and the length L of the cable.

(b) Assume that the hangers are weightless and that the constant external distributed load is given by

$$q_L^* = q_L + q_c = 1.1 \cdot 10^5 + 4.621 \cdot 10^4 = 1.56 \cdot 10^5 \text{ N/m},$$

where q_c is the weight per unit length of the cable. Compute H, the maximum cable tension $T_{\max}$ and the length L of the cable.

Solution (a)

First, the location of the origin of the co-ordinate system Oxz is chosen such that

$$(x_A, z_A) = (0, 150) \text{ m}, \quad (x_B, z_B) = (1300, 150) \text{ m} \quad \text{and} \quad (x_D, z_D) = (650, 0) \text{ m},$$

where x_D is calculated as

$$x_D = \frac{x_A + x_B}{2} = 650 \text{ m}.$$

Then, by inserting q_L, x_B, z_B, x_D and z_D into Equation (3.10), we obtain the equation that is solved for H to give

$$H = 1.549 \cdot 10^8 \text{ N}.$$

The maximum cable tension occurs at both ends of the cable where, according to Equation (3.15), we get

$$T_{\max} = f_T(x_A) = f_T(x_B) = 1.706 \cdot 10^8 \text{ N}.$$

The length of the cable L is calculated numerically by using Equation (3.16), which yields

$$L = f_s(x_B) = 1344.781 \text{ m}.$$

Solution (b)

By replacing q_L with q_L^*, and then proceed in the same way as in Part (a) of this example, we obtain

$$H = 2.200 \cdot 10^8 \text{ N},$$

$$T_{\max} = f_T(x_A) = f_T(x_B) = 2.423 \cdot 10^8 \text{ N},$$

$$L = f_s(x_B) = 1344.781 \text{ m}.$$

It is seen that the length of the inextensible cable is not affected by the value of the constant load distribution.

The shape of the cable is shown in Figure E3.1.1.

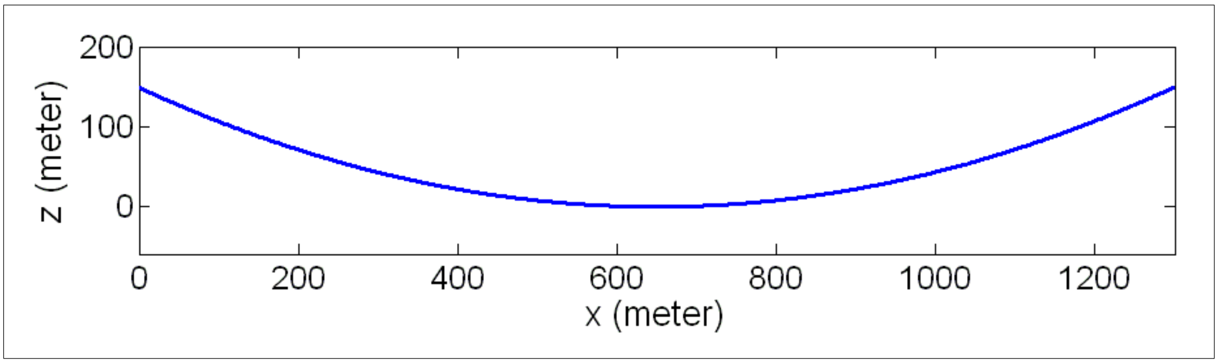


Figure E3.1.1: The shape of the main span section of the main cable.

Example 3.2, I

We assume that the bridge described in Example 3.1 is of Type 2, as shown in Figure 3.3, and that the side span sections of the bridge deck are of length 390 m. It is desirable that the magnitude of the horizontal component H of the cable tension, in each main cable, is the same on both sides of the towers, as shown in Figure E3.2.1 below.

Points A and B of section III of the main cables are, respectively, assumed to have the same x co-ordinate as the ends of the underlying side span section of the bridge deck. Section III of each main cable is assumed to carry half the weight of the portion of the bridge deck that stretches horizontally between x_A and x_B.

We shall calculate the maximum cable tension, and the length of Section III of each main cable, assuming that $H = 2.20998 \cdot 10^8$ N, and that the distributed load is equal to q_L^* of Example 3.1. The value of H given above is calculated in Example 3.1, III in Chapter 7.

The same co-ordinate system as in Example 3.1 is used, and we assume that the co-ordinates of the end points of the cable are as follows:

$$(x_A, z_A) = (1310, 150) \text{ m} \quad \text{and} \quad (x_B, z_B) = (1700, 2) \text{ m}.$$

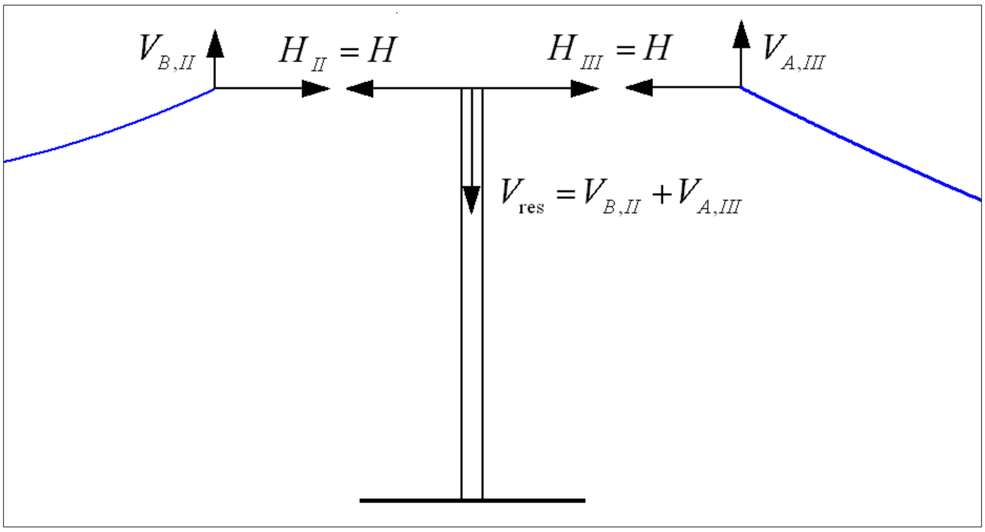


Figure E3.2.1: Forces acting on the ends of different sections of one of the main cables, and their reactions on the tower. Ideally, the horizontal forces acting on the tower are equal in magnitude and opposite in direction, which means that $H_{II} = H_{III} = H$. The subscripts II and III refer to different sections of the main cable, which are defined in Figure 3.3.

Solution

As it is required that H is to assume the same value in all sections of the cable, we solve the linear system of equations (3.12) for C_1 and C_2, which gives

$$C_1 = -1.443 \quad \text{and} \quad C_2 = 1.434 \cdot 10^3 \text{ m}.$$

The co-ordinate x_D is now calculated as

$$x_D = -\frac{C_1 H}{q_L} = 2042 \text{ m}.$$

The shape of the cable is shown in Figure E3.2.2, which is located at the end of this chapter.

According to Equation (3.15), the maximum cable tension occurs at point A, and it is equal to

$$T_{\max} = f_T(x_A) = 2.488 \cdot 10^8 \text{ N}.$$

Finally, the length of the cable is given by Equation (3.16):

$$L = f_s(x_B) = 418.146 \text{ m}.$$

Example 3.3, I

In this example, we analyse the main span section of each main cable of an asymmetric suspension bridge. The cable has a span of $d_h = 1164$ m, and a sag of $d_v = 200$ m, when no vehicles are on the bridge. Points A and B are, respectively, located at

$$(x_A, z_A) = (0, 100) \text{ m} \quad \text{and} \quad (x_B, z_B) = (1164, 200) \text{ m},$$

whereas the z co-ordinate of point D is required to be equal to

$$z_D = 0.$$

The road deck is horizontal, and its constant weight per unit length along the span is $q_{L,\text{total}} = 3 \cdot 10^5$ N/m. Each main cable is assumed to carry half the weight of the portion of the bridge deck that stretches horizontally between x_A and x_B, which means that

$$q_L = \frac{q_{L,\text{total}}}{2} = 1.5 \cdot 10^5 \text{ N/m}.$$

It is assumed that the towers are rigid and that the main cables cannot move relative to the towers.

Neglect the weight of the hangers, and assume that the constant external distributed load is given by

$$q_L^* = q_L + q_c = 1.5 \cdot 10^5 + 5.391 \cdot 10^4 = 2.039 \cdot 10^5 \text{ N/m},$$

where q_c is the weight per unit length of the cable.

Compute H, the maximum cable tension $T_{\max}$ and the length L of the cable.

Solution

By replacing q_L by q_L^*, the system of equations (3.11) is solved numerically to give

$$H = 2.370 \cdot 10^8 \text{ N} \quad \text{and} \quad x_D = 482.145 \text{ m}.$$

The starting values for the numerical solution were obtained graphically.

According to Equation (3.15), the maximum cable tension is found at point B (see also Figure E3.3.2):

$$T_{\max} = f_T(x_B) = 2.748 \cdot 10^8 \text{ N}.$$

Equation (3.16) gives the length of the cable as

$$L = f_s(x_B) = 1214.794 \text{ m}.$$

The shape of the cable is shown in Figure E3.3.1.

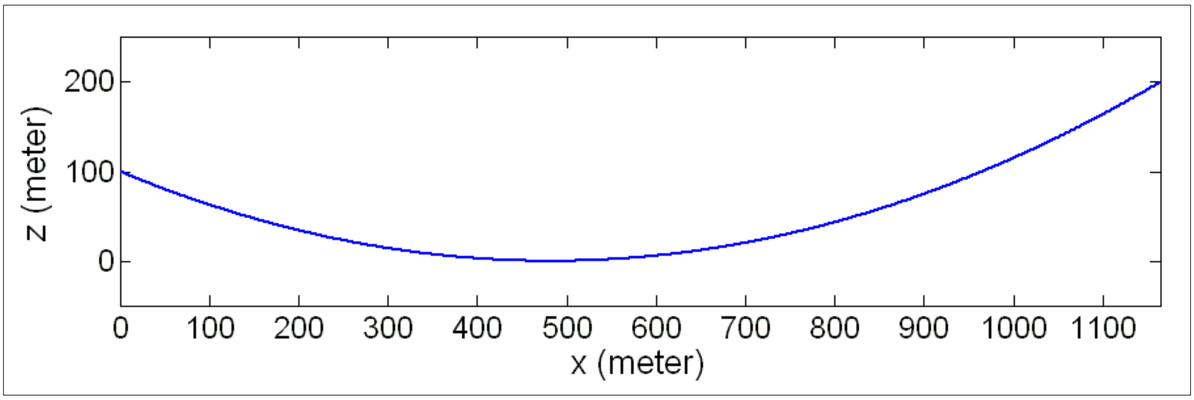


Figure E3.3.1: The shape of the main span section of the main cable.

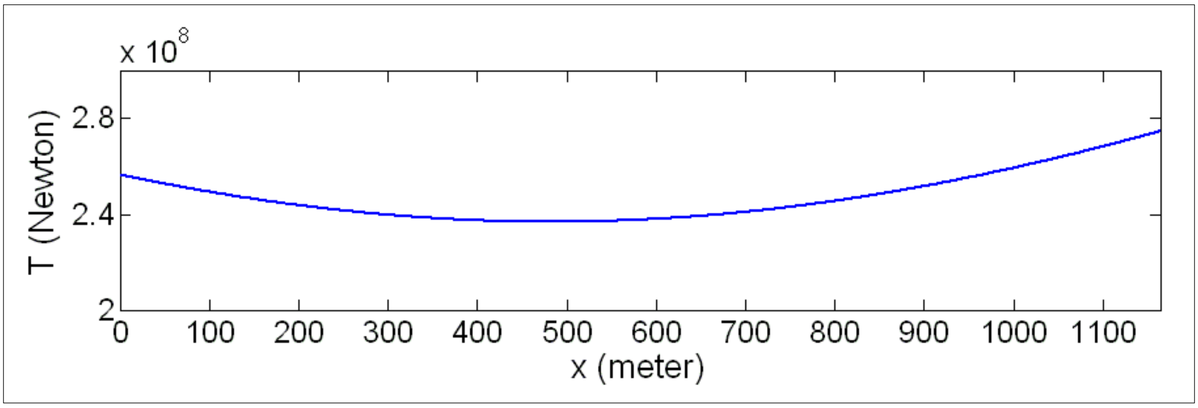


Figure E3.3.2: The cable tension $T = f_T(x)$ versus the co-ordinate x.

Figure E3.2.2 of Example 3.2, I

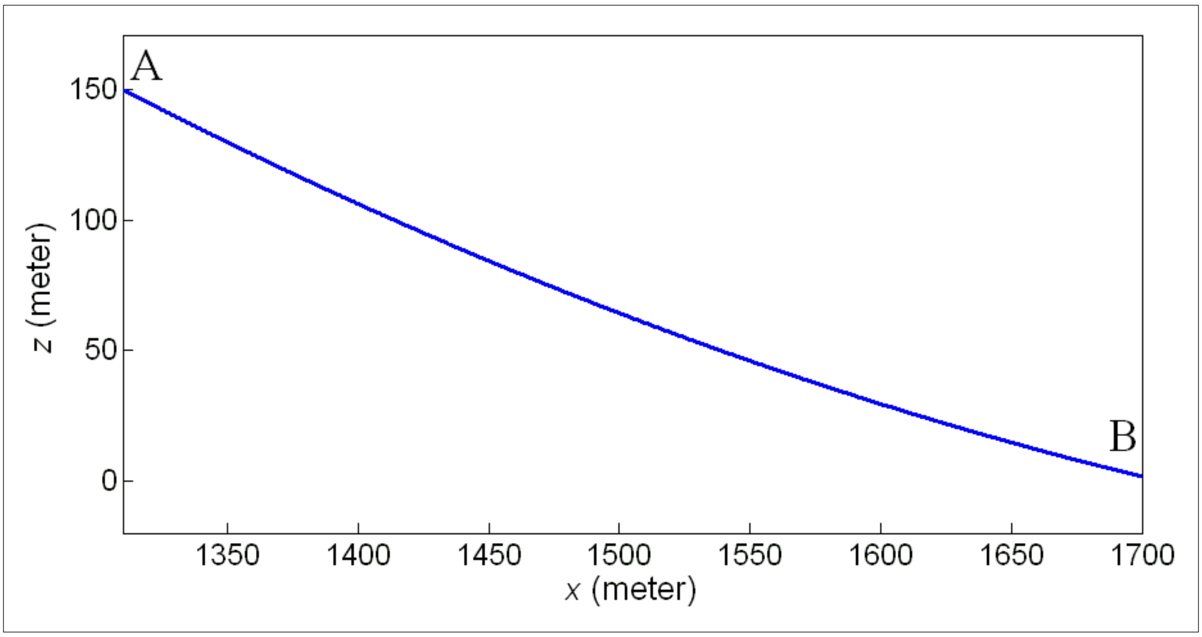


Figure E3.2.2: The shape of the side span section of the main cable.

4. The linearly elastic catenary

For cables with deep sag and high axial rigidity, the previously described theory of the inextensible catenary will usually produce rather accurate results. In reality, no cables are inextensible and, in many technically important applications, the sag of the cable is required to be quite shallow. This infers that the cable tension can be sufficiently high to cause considerable elongations of cables that are under the action of gravity only. Examples of such cables are the main cables of suspension bridges, long and heavy suspended electrical cables and cables of low axial rigidity. It is therefore obvious that theories of inextensible cables are insufficient in some cases. There are several different theories of elastic cables, some of which are only applicable to cables of small to moderate sags. In this chapter, we describe a theory that, within the validity of its assumptions, is an exact theory of the linearly elastic catenary. The theory is presented in Irvine and Sinclair [2], and, as is shown in later chapters of the present paper, this theory can be extended to include additional external vertical point forces, or be otherwise modified.

In the theory of the linearly elastic catenary described here, the length L_0 of the unstrained cable is assumed to be initially known. If, as before, it is required that the cable under self weight is to assume a shape associated with a prescribed span d_h and sag d_v, then the length of the unstrained cable has to be calculated. So far, the only equation we have derived for the length of a cable that is under the action of gravitational load only, is Equation (2.17), applicable for inextensible cables. However, an extensible cable will elongate when it is loaded, and therefore, the sag of an elastic cable will deviate from the prescribed value if the initial length is given by Equation (2.17). In many situations, the elongation of the cable will be quite small, but in situations where the assumption of inextensibility is insufficient, the initial length of the cable must be calculated in a different way. In Chapter 5 we will show how to calculate the initial length of the cable such that the sag of the linearly elastic cable, described in this chapter, is equal to a prescribed value.

It is assumed that the unstrained cable is of constant cross sectional area A_0, and of constant weight per unit length $q_c = m_0 g$, where m_0 is the mass per unit length of the unstrained cable, and g is the gravitational acceleration. Hence, the self weight of the entire cable is equal to $W = q_c L_0$.

As shown in Figure 4.1 below, the ends of the cable centerline are located at the fixed points A and B, respectively.

An arc-length co-ordinate s, $0 \leq s \leq L_0$, is introduced along the centerline of the unstrained cable. The position co-ordinates of a point on the centerline of the strained cable are assumed to be given by the Cartesian co-ordinates $x = f_x(s)$ and $z = f_z(s)$, relative to the co-ordinate system Oxz, as well as by the arc-length co-ordinate $p = f_p(s)$, along the centerline of the strained cable. The location of the origin of the co-ordinate system Oxz may be chosen arbitrarily.

We require that the functions $f_x(s)$, $f_z(s)$ and $f_p(s)$ satisfy the following boundary conditions:

$$x_A = f_x(0) \quad \text{(a)}, \qquad z_A = f_z(0) \quad \text{(b)}, \qquad p_A = f_p(0) = 0 \quad \text{(c)},$$

$$x_B = f_x(L_0) \quad \text{(d)}, \qquad z_B = f_z(L_0) \quad \text{(e)}, \qquad p_B = f_p(L_0) = L \quad \text{(f)}, \qquad (4.1)$$

where L is the initially unknown length of the strained cable, (x_A, z_A) and (x_B, z_B) are the position co-ordinates of points A and B, respectively.

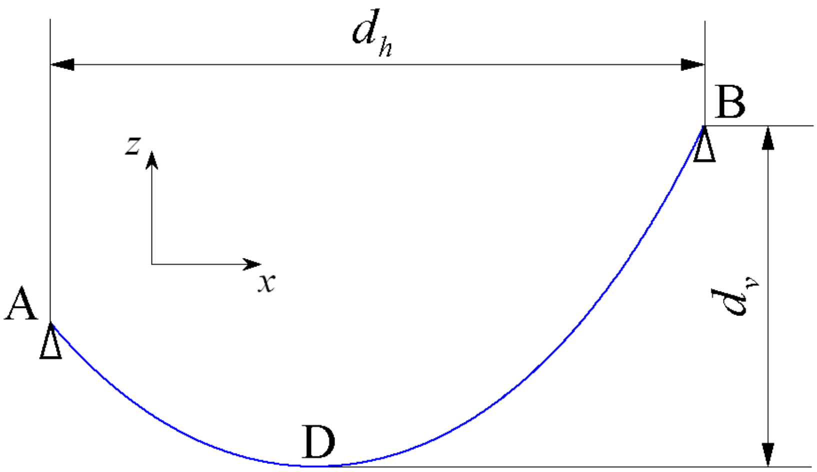


Figure 4.1: The ends of the cable are rigidly supported at points A and B. In the figure is also shown the span d_h and sag d_v of the cable. We may choose the location of the origin of the co-ordinate system Oxz arbitrarily. The x axis is horizontal, and the z axis is vertical.

The following assumptions regarding the properties of the cable are also made:

- Due to the principle of conservation of mass, any portion of the unstrained cable, located between two planar cross-sections which are perpendicular to the centerline of the cable, will retain its mass during deformation. Hence, the weight of the portion of the strained cable shown in Figure 4.2, can be written as $W\dfrac{s}{L_0} = q_c s$.

- The cable has no flexural rigidity.

- The only stress that occurs in the cable is the axial normal tensile stress $\sigma_N = f_{\sigma_N}(s)$, which is uniformly distributed over each cross-section perpendicular to the centerline of the cable.

- In the axial direction, the cable behaves as a homogenous linearly elastic material that undergoes infinitesimal strains. Therefore, the equation for the axial strain ε is given by

$$\varepsilon = f_\varepsilon(s) = \frac{dp - ds}{ds} = \frac{dp}{ds} - 1. \tag{4.2}$$

By using Equation (4.2) together with the assumption that the cross-sectional normal stress σ_N is uniformly distributed over each cross-section, the cable tension T can be written as

$$T = f_T(s) = \sigma_N A_0 = E\varepsilon A_0 = EA_0\left(\frac{dp}{ds} - 1\right), \tag{4.3}$$

where E is the constant modulus of elasticity, and EA_0 is the constant axial rigidity of the cable. It is then recalled that the reduction of the cross-sectional area A_0, due to $\sigma_N = f_{\sigma_N}(s)$, is neglected.

As shown in Figure (4.2), the co-ordinates defined above imply that

$$\frac{dz}{dp} = \sin(\theta) \qquad \text{(a)}, \qquad \frac{dx}{dp} = \cos(\theta) \qquad \text{(b)}, \tag{4.4}$$

where $\theta = f_\theta(s)$, $-\pi/2 \leq \theta \leq \pi/2$, is the angle of inclination of the cable at p.

Horizontal and vertical equilibrium of the portion of the strained cable shown in Figure 4.2 requires, respectively, that

$$T\frac{dx}{dp} = H, \tag{4.5}$$

$$T\frac{dz}{dp} = q_c s - V_A, \tag{4.6}$$

where H is the constant horizontal component of the cable tension $T = f_T(s)$, and V_A is the scalar component of the vertical reaction force $\boldsymbol{V_A}$ at point A.

In order to find the equation for the cable tension T, Equations (4.5) and (4.6) are rewritten as

$$\frac{dx}{dp} = \frac{H}{T}, \tag{4.7}$$

$$\frac{dz}{dp} = \frac{1}{T}\left(q_c s - V_A\right). \tag{4.8}$$

Insertion of Equations (4.7) and (4.8) into the geometry relation

$$\left(\frac{dx}{dp}\right)^2 + \left(\frac{dz}{dp}\right)^2 = 1 \tag{4.9}$$

yields

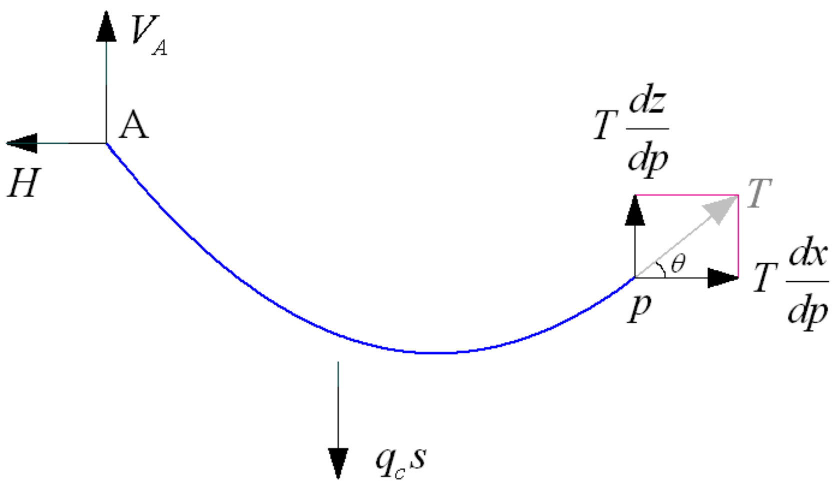


Figure 4.2: Forces acting on a portion of the strained cable. The length of the shown portion is $p = f_p(s)$, and its self weight is $q_c s$.

$$T = f_T(s) = \sqrt{H^2 + (q_c s - V_A)^2}. \tag{4.10}$$

Our next objective is to find the equation for the position co-ordinate $x = f_x(s)$. For this reason, we write

$$\frac{dx}{ds} = \frac{dx}{dp}\frac{dp}{ds}. \tag{4.11}$$

By inserting T, as given by Equation (4.10), into Equation (4.7), we obtain

$$\frac{dx}{dp} = \frac{H}{T} = \frac{H}{\sqrt{H^2 + (q_c s - V_A)^2}}. \tag{4.12}$$

The derivative dp/ds is calculated by using Equations (4.2), (4.3) and (4.10) according to

$$\frac{dp}{ds} = \varepsilon + 1 = \frac{T}{EA_0} + 1 = \frac{1}{EA_0}\sqrt{H^2 + (q_c s - V_A)^2} + 1. \tag{4.13}$$

Insertion of Equations (4.12) and (4.13) into Equation (4.11), and subsequent integration, yields

$$\int dx = \int \frac{H}{EA_0} ds + \int \frac{H}{\sqrt{H^2 + (q_c s - V_A)^2}} ds. \tag{4.14}$$

By carrying out the integrals, we get

$$x = \frac{H}{EA_0} s + \frac{HL_0}{W} \sinh^{-1}\left(\frac{q_c s - V_A}{H}\right) + C, \tag{4.15}$$

where C is a constant of integration. It is required that Equation (4.15) satisfies the condition $x_A = f_x(0)$. With this information, we calculate C by inserting $s = 0$ and $x = x_A$ into Equation (4.15), which is then solved for C. Once C is known, the equation for x can be written as

$$x = f_x(s) = x_A + \frac{H}{EA_0} s + \frac{H}{q_c}\left(\sinh^{-1}\left(\frac{q_c s - V_A}{H}\right) - \sinh^{-1}\left(-\frac{V_A}{H}\right)\right). \tag{4.16}$$

We will now derive the equation for the position co-ordinate $z = f_z(s)$. It holds that

$$\frac{dz}{ds} = \frac{dz}{dp}\frac{dp}{ds}. \tag{4.17}$$

From Equations (4.6) and (4.10) we get

$$\frac{dz}{dp} = \frac{1}{T}(q_c s - V_A) = \frac{q_c s - V_A}{\sqrt{H^2 + (q_c s - V_A)^2}}. \tag{4.18}$$

Insertion of Equations (4.18) and (4.13) into to Equation (4.17), multiplication by ds, and subsequent integration, result in

$$\int dz = \int \frac{1}{EA_0}\left(q_c s - V_A\right) ds + \int \frac{q_c s - V_A}{\sqrt{H^2 + \left(q_c s - V_A\right)^2}} ds. \tag{4.19}$$

By performing the integrals in Equation (4.19), we get

$$z = \frac{1}{EA_0}\left(\frac{q_c}{2}s^2 - V_A s\right) + \frac{1}{q_c}\sqrt{H^2 + \left(q_c s - V_A\right)^2} + C, \tag{4.20}$$

where C is a constant of integration. We require that Equation (4.20) satisfies the boundary condition $z_A = f_z(0)$. With the aid of this boundary condition, we calculate C by inserting $s = 0$ and $z = z_A$ into Equation (4.20), which is then solved for C. The equation for z now takes the form

$$z = f_z(s) = z_A + \frac{1}{EA_0}\left(\frac{q_c}{2}s^2 - V_A s\right) + \frac{1}{q_c}\left(\sqrt{H^2 + \left(q_c s - V_A\right)^2} - \sqrt{H^2 + {V_A}^2}\right). \tag{4.21}$$

If the values of E, A_0, q_c and L_0 are known, then there are two unknown constants, H and V_A, that must be determined before the solution is complete. By inserting $s = L_0$ and $x = x_B$ into Equation (4.16), and inserting $s = L_0$ and $z = z_B$ into Equation (4.21), we obtain

$$0 = -x_B + x_A + \frac{HL_0}{EA_0} + \frac{H}{q_c}\left(\sinh^{-1}\left(\frac{q_c L_0 - V_A}{H}\right) - \sinh^{-1}\left(-\frac{V_A}{H}\right)\right) \tag{4.22a}$$

$$0 = -z_B + z_A + \frac{L_0}{EA_0}\left(\frac{q_c L_0}{2} - V_A\right) + \frac{1}{q_c}\left(\sqrt{H^2 + \left(q_c L_0 - V_A\right)^2} - \sqrt{H^2 + {V_A}^2}\right). \tag{4.22b}$$

This system of equations is solved numerically for H and V_A. The starting values H_0 and $V_{A,0}$, necessary to initiate the numerical solution procedure, can be obtained by plotting, in the same figure, the plane given by the function $f_0(H, V_A) = 0$, together with the function graphs of the two functions

$$f_1(H, V_A) = -x_B + x_A + \frac{HL_0}{EA_0} + \frac{H}{q_c}\left(\sinh^{-1}\left(\frac{q_c L_0 - V_A}{H}\right) - \sinh^{-1}\left(-\frac{V_A}{H}\right)\right) \tag{4.23a}$$

$$f_2(H, V_A) = -z_B + z_A + \frac{L_0}{EA_0}\left(\frac{q_c L_0}{2} - V_A\right) + \frac{1}{q_c}\left(\sqrt{H^2 + \left(q_c L_0 - V_A\right)^2} - \sqrt{H^2 + {V_A}^2}\right), \tag{4.23b}$$

which are obtained from Equations (4.22). Useful values of H_0 and $V_{A,0}$ exist in the vicinity of the physically relevant point of intersection of the graphs given by f_0, f_1 and f_2.

If the supports at the ends of the cable are on the same vertical level, we have that $z_A = z_B$ and, consequently, the shape of the cable is symmetric. This infers that $V_A = V_A^{\mathrm{sym}} = W/2 = q_c L_0/2$. It can be shown that V_A^{sym} is a solution to Equation (4.22b), and that this solution is independent of H. We therefore determine the horizontal component of the cable tension by numerically solving Equation (4.22a) for H. This equation then takes the form

$$0 = -x_B + x_A + \frac{HL_0}{EA_0} + \frac{2H}{q_c} \sinh^{-1}\left(\frac{q_c L_0}{2H}\right). \tag{4.24}$$

The length of the strained cable is calculated as

$$L = f_p(L_0) = \int_0^{L_0} \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2} ds, \tag{4.25}$$

where

$$\frac{dx}{ds} = \frac{df_x}{ds}(s) = \frac{H}{EA_0} + \frac{1}{\sqrt{1 + ((q_c s - V_A)/H)^2}}, \tag{4.26}$$

$$\frac{dz}{ds} = \frac{df_z}{ds}(s) = \frac{q_c s - V_A}{EA_0} + \frac{q_c s - V_A}{\sqrt{H^2 + (q_c s - V_A)^2}}. \tag{4.27}$$

Evaluation of the integral (4.25) is done numerically in the present paper.

When all parameters of Equations (4.16) and (4.21) are known, we consider the functions f_x and f_z to be functions of the arc-length coordinate s, that is $x = f_x(s)$ and $z = f_z(s)$. However, in some situations, the functions f_x and f_z may be considered to be functions of several variables. By adopting this approach, we can conveniently write Equations (4.22) as

$$0 = -x_B + f_x(H, V_A)|_{s=L_0} \tag{4.28a}$$

$$0 = -z_B + f_z(H, V_A)|_{s=L_0}, \tag{4.28b}$$

where the notation $|_{s=L_0}$ means that $s = L_0$ in the equations.

Programming of Equations (4.16) and (4.21), for problems that involve solution of a system of nonlinear equations, may be simplified if an additional co-ordinate system $Ox_c z_c$, is introduced (see Figure 4.3 below). This co-ordinate system has its origin at point A, and the position co-ordinates of the centerline of the strained cable are $x_c = f_{x_c}(s)$ and $z_c = f_{z_c}(s)$, relative to the co-ordinate system $Ox_c z_c$. As seen in Figure (4.3), the relation between x and x_c is given by

$$x = x_A + x_c, \tag{4.29}$$

whereas the relation between z and z_c reads

$$z = z_A + z_c. \tag{4.30}$$

With this approach, equations (4.16) and (4.21), respectively, takes the form

$$x = f_x(s) = x_A + f_{x_c}(s), \tag{4.31}$$

$$z = f_z(s) = z_A + f_{z_c}(s). \tag{4.32}$$

This means that the system of Equations (4.26) can be written as

$$0 = -x_{cB} + f_{x_c}(H, V_A)|_{s=L_0} \tag{4.33a}$$

$$0 = -z_{cB} + f_{z_c}(H, V_A)|_{s=L_0}, \tag{4.33b}$$

where

$$x_{cB} = x_B - x_A, \quad z_{cB} = z_B - z_A. \tag{4.34}$$

One would then create programs for the functions f_{x_c} and f_{z_c}, instead of for the functions f_x and f_z. This method is often used in the remainder of the present paper, and it is particularly useful in Chapters 5 and 7, where we solve problems involving a large number of equations.

In the examples of this chapter, we compare results obtained under the assumption of axial inextensibility, with results obtained assuming that the cable is axially elastic. Therefore, in order to facilitate comparison of results calculated under different assumptions, we use in the subsequent examples of the present paper the subscript ie, which stands for inextensible, in order to indicate which results are calculated under the assumption of axial inextensibility.

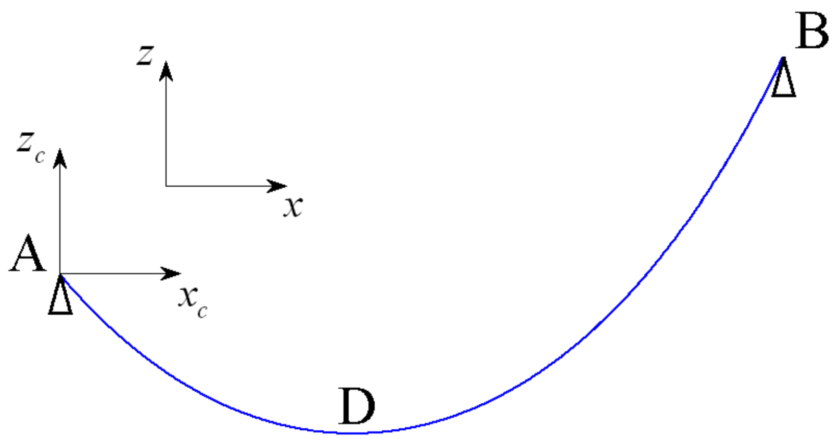


Figure 4.3: While the origin of the co-ordinate system Oxz may be given an arbitrary location, the origin of the co-ordinate system Ox_cz_c is always located point A.

Example 2.1, II (Continued from Chapter 2)

We will now analyse the cable according to the theory of the linearly elastic catenary. The unstrained cable, whose length is taken to be $L_0 = L_{\mathrm{ie}}$, is of cross-sectional area $A_0 = 5 \cdot 10^{-4}$ m^2, and the modulus of elasticity in the axial direction of the cable is $E = 1.5 \cdot 10^{11}$ N/m^2. Hence, the axial rigidity of the cable is $EA_0 = 7.5 \cdot 10^7$ N.

Calculate H, V_A, the z co-ordinate of the lowest point of the cable centerline, the length L of the strained cable and the elongation ΔL of the cable.

For convenience, the results of the analysis performed in the first part of this example (Example 2.1, I) are repeated here:

$$H_{\mathrm{ie}} = 2.5013 \cdot 10^4 \text{ N}, \quad L_{\mathrm{ie}} = 100.107 \text{ m}.$$

Solution

Owing to the fact that the cable is of symmetric shape, we can calculate V_A as

$$V_A = \frac{q_c L_0}{2} = 2002 \text{ N}.$$

Equation (4.24) is now solved to give

$$H = 2.2125 \cdot 10^4 \text{ N}.$$

As starting value for the numerical solution, we used $H_0 = 2 \cdot 10^4$ N, which was obtained graphically. Figure E2.1.2 shows the convergence behaviour of the numerical calculation of H.

Next, we use Equation (4.21) to calculate the z co-ordinate of the lowest point of the cable centerline:

$$z_D = f_z(L_0/2) = -2.261 \text{ m}.$$

For comparison, we compute the difference between the co-ordinate z_D of the elastic cable, and the co-ordinate $z_{D,\mathrm{ie}} = -2$ m, of the inextensible cable:

$$z_D - z_{D,\mathrm{ie}} = -0.261 \text{ m}.$$

This means that the sag of the elastic cable is noticeably deeper than that of the inextensible cable.

Equation (4.25) gives the length of the strained cable, that is

$$L = 100.136 \text{ m}.$$

The elongation of the cable is calculated as the difference between the length of the strained cable and the length of the unstrained cable:

$$\Delta L = L - L_0 = 0.0296 \text{ m}.$$

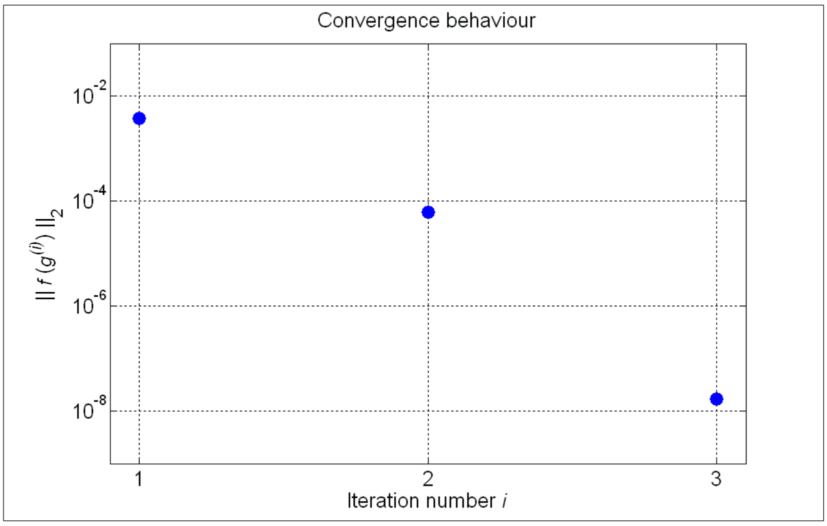


Figure E2.1.2: Convergence behaviour of the numerical calculation of H. The number of iterations is actually 4, but that is not shown since $||f(g^{(4)})||_2 = 0$.

Example 2.2, II (Continued from Chapter 2)

The cable is here analysed according to the theory of the linearly elastic catenary. The cross-sectional area of the unstrained cable is $A_0 = 5 \cdot 10^{-4}$ m^2, and the modulus of elasticity is $E = 1.5 \cdot 10^{11}$ N/m^2, in the axial direction. Consequently, the axial rigidity of the cable is $EA_0 = 7.5 \cdot 10^7$ N. Assuming that the length of the unstrained cable is $L_0 = L_{\text{ie}}$, calculate H, V_A, the x and z co-ordinate of the lowest point of the cable centerline, the length L of the strained cable and the elongation ΔL of the cable.

The results of the analysis performed in the first part of this example are repeated here:

$$H_{\text{ie}} = 1.7178 \cdot 10^4 \text{ N}, \quad L_{\text{ie}} = 100.246 \text{ m}.$$

Solution

The shape of the cable is asymmetric and, therefore, we cannot calculate the value of V_A directly as we did in the previous example. Instead, H and V_A are determined by numerically solving the system of equations (4.22). The starting values for the numerical solution are determined graphically, which gives $H_0 = 1.6 \cdot 10^4$ N and $V_{A,0} = 1700$ N. Solving the system of Equations (4.22) yields

$$H = 1.6397 \cdot 10^4 \text{ N}, \quad V_A = 1675 \text{ N}.$$

Figure E2.2.7 shows the convergence behaviour of the numerical calculation of H and V_A.

Since the shape of the cable is asymmetric, we do not know the co-ordinates s_D and x_D of the lowest point of the cable centerline. However, s_D can be determined by solving the equation

$$\frac{dz}{ds} = 0$$

for s, which gives

$$s_D = 41.884 \text{ m.}$$

Insertion of s_D into Equations (4.16) and (4.21), yields

$$x_D = f_x(s_D) = 166.821 \text{ m},$$

$$z_D = f_z(s_D) = 23.865 \text{ m}.$$

By calculating the difference between the co-ordinate z_D of the elastic cable, and the co-ordinate $z_{D,\text{ie}} = 24$ m, of the inextensible cable, we get

$$z_D - z_{D,\text{ie}} = -0.135 \text{ m}.$$

The shape of the cable is shown in Figure E2.2.6.

Equation (4.25) gives the length of the strained cable, that is

$$L = 100.268 \text{ m.}$$

The elongation of the cable is calculated as

$$\Delta L = L - L_0 = 0.0220 \text{ m}.$$

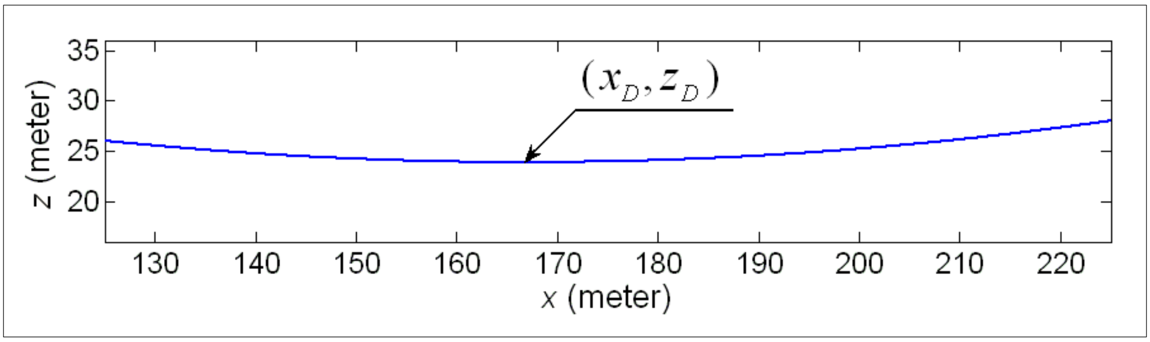


Figure E2.2.6: The shape of the cable.

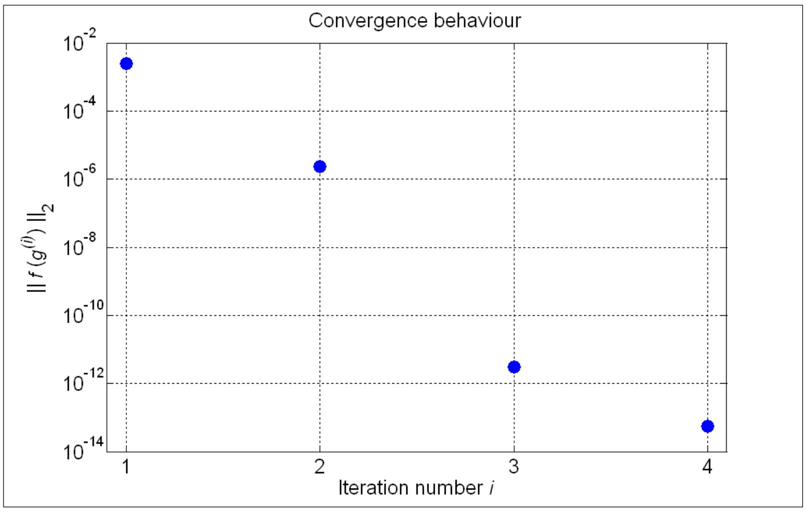


Figure E2.2.7: Convergence behaviour of the numerical calculation of H and V_A.

5. Shape control of the linearly elastic catenary

The theory of the linearly elastic catenary, described in the previous chapter, requires that the initial length of the cable L_0 is known. As shown in Examples 2.1 and 2.2, there are situations where the shape of an elastic cable cannot be accurately controlled if the initial length of the cable is calculated according to the theory of the inextensible catenary.

The present chapter deals basically with two types of problems that involve computation of the length of an unstrained linearly elastic cable, in such a way that the desired results can be obtained.

5.1 The linearly elastic catenary with prescribed sag d_v

In situations where the span d_h and sag d_v of a cable are required to assume prescribed values, the length of the cable is initially unknown. It may, for instance, be necessary to ensure that the height above the ground of the lowest point of a suspended electrical cable is not less than a certain allowable minimum height in the static configuration. At the same time, it is desirable to keep the height of the towers that carry the cables as low as possible in order to reduce the amount of material used. By using a shorter cable, the sag d_v is likely to decrease. However, a decrease of the sag will usually cause the cable tension to increase, which, in turn, may require a stronger and possibly more expensive cable. Assuming that the co-ordinates of the end points of the cable centerline are prescribed, it may, therefore, be advantageous to be able to calculate the length of the cable such that the height above the ground of its lowest point is equal to the minimum allowable height. As was seen in Examples 2.1 and 2.2, the theory of the inextensible catenary is in some cases not able to accurately predict the shape of an elastic cable. In this section, we show that the theory of the linearly elastic catenary can be used in order to determine the length L_0 of an unstraned cable, such that the span d_h and sag d_v are equal to prescribed values. We then consider the case where (cf. Figure 2.1 on page 8)

$$d_v > \|z_B - z_A\|_2 \,, \tag{5.1}$$

which implies that the lowest point of the cable center line, located at

$$(x_D, z_D) = (f_x(s_D), f_z(s_D)), \quad x_A < x_D < x_B \tag{5.2}$$

is also the point with horizontal tangent vector. Consequently, we have that

$$\frac{dz}{ds} = \frac{df_z}{ds}(s_D) = 0. \tag{5.3}$$

In a situation where the shape of the cable is symmetric, it holds that

$$x_D = \frac{x_B + x_A}{2} \quad \text{and} \quad s_D = \frac{L_0}{2},$$

but if the cable is of asymmetric shape, both s_D and x_D must be calculated. In either case, the co-ordinate z_D is prescribed such that the condition

$$z_D < \min(z_A, z_B) \tag{5.4}$$

is fulfilled, which is in agreement with condition (5.1).

Boundary conditions (4.1) remain unchanged since translational motion of the ends of the cable is prevented.

The shape of the cable is, as before, given by Equations (4.16) and (4.21), but these equations alone are insufficient to determine the unknown parameters H, V_A, L_0 and, in case of an asymmetric cable, also s_D. Therefore, additional equations are added to the system of Equations (4.33) so that, in the case of a cable of symmetric shape, we get

$$\begin{aligned} 0 &= -x_{cB} + f_{x_c}(H, V_A, L_0)|_{s=L_0} \\ 0 &= -z_{cB} + f_{z_c}(H, V_A, L_0)|_{s=L_0} \\ 0 &= -z_{cD} + f_{z_c}(H, V_A, L_0)|_{s=L_0/2}, \end{aligned} \tag{5.5}$$

where

$$x_{cB} = x_B - x_A, \quad z_{cB} = z_B - z_A, \quad z_{cD} = z_D - z_A.$$

Equations (5.5) are solved for H, V_A and L_0. Note that since the shape of the cable is symmetric, we have that $z_A = z_B$, which infers that $z_{cB} = 0$.

In a problem where the shape of the cable is asymmetric, the unknown parameters H, V_A, L_0 and s_D are determined by solving a system of equations that is given by

$$\begin{aligned} 0 &= -x_{cB} + f_{x_c}(H, V_A, L_0)|_{s=L_0} \\ 0 &= -z_{cB} + f_{z_c}(H, V_A, L_0)|_{s=L_0} \\ 0 &= -z_{cD} + f_{z_c}(H, V_A, s_D)|_{s=s_D} \\ 0 &= \frac{df_z}{ds}(H, V_A, s_D)|_{s=s_D}, \end{aligned} \tag{5.6}$$

where

$$\frac{df_z}{ds} = \frac{1}{EA_0}(q_c s - V_A) + \frac{q_c s - V_A}{\sqrt{H^2 + (q_c s - V_A)^2}}. \tag{5.7}$$

The parameters H_0, $V_{A,0}$, $L_{0,0}$ and $s_{D,0}$, necessary to start the numerical solution process, can usually be supplied by the previously described theory of the inextensible catenary.

Once the systems of equations (5.5) and (5.6) are solved, we compare the results provided by the functions $x = f_x(s)$ and $z = f_z(s)$, with the prescribed values that these functions are required to provide at certain values of the arc-length co-ordinate s. This is done by calculating the difference between the prescribed values of the Cartesian co-ordinates x and z, which are here, respectively, denoted x^{pres} and z^{pres}, and the values obtained from the functions $x = f_x(s)$ and $z = f_z(s)$, according to

$$x^{\text{diff}} = x^{\text{pres}} - f_x(s), \tag{5.8}$$

$$z^{\text{diff}} = z^{\text{pres}} - f_z(s). \tag{5.9}$$

This method for comparing results is also used in Chapter 7 of the present paper.

5.2 The linearly elastic catenary with prescribed value of H

As described in Chapter 3, it is desirable that the magnitude of the horizontal component H of the cable tension T, in the main cables of a suspension bridge, is the same on both sides of the towers (see Figure E3.2.1). For each side span section of the main cables, this can be accomplished for a prescribed value of H, and prescribed values of the co-ordinates (x_A, z_A) and (x_B, z_B) of the end points of the cable section, by solving the following system of equations for V_A and L_0:

$$\begin{aligned} 0 &= -x_{cB} + f_{x_c}(V_A, L_0)|_{s=L_0} \\ 0 &= -z_{cB} + f_{z_c}(V_A, L_0)|_{s=L_0}, \end{aligned} \tag{5.10}$$

where

$$x_{cB} = x_B - x_A, \qquad z_{cB} = z_B - z_A.$$

Starting values for the numerical solution can be obtained graphically.

Example 2.1, III (Continued from Chapter 4)

Calculate H, V_A and the initial length L_0 of the cable such that

$$x_D^{\text{req}} = f_x(s_D) = f_x(L_0/2) = 50 \text{ m}, \tag{E.2.1.2}$$

$$z_D^{\text{req}} = f_z(s_D) = f_z(L_0/2) = -2 \text{ m}. \tag{E.2.1.3}$$

Also calculate the length L of the strained cable and the elongation ΔL of the cable.

The co-ordinates of the end points of the cable are, respectively, equal to

$$(x_A, z_A) = (0, 0) \text{ m} \quad \text{and} \quad (x_B, z_B) = (100, 0) \text{ m}.$$

For convenience, the results of the analysis performed in the first part of this example are repeated here:

$$H_{\text{ie}} = 25013 \text{ N}, \quad L_{\text{ie}} = 100.107 \text{ m}.$$

Solution

As the shape of the cable is symmetric, we solve the system of equations (5.5) for H, V_A and L_0. For the numerical solution process, we use the starting values

$$H_0 = H_{\text{ie}}, \quad L_{0,0} = L_{\text{ie}}, \quad V_{A,0} = \frac{q_c L_{\text{ie}}}{2} = 2002 \text{ N}.$$

The result of the numerical calculations is given by

$$H = 2.5005 \cdot 10^4 \text{ N}, \quad V_A = 2001 \text{ N}, \quad L_0 = 100.073 \text{ m}.$$

Figure E2.1.3, which is placed at the end of this chapter, shows the convergence behaviour of the numerical calculation of H, V_A and L_0.

In order to find out how well Equations (4.16) and (4.21) satisfies conditions (E.2.1.2) and (E.2.1.3), respectively, we compute x_D^{diff} and z_D^{diff} according to Equations (5.8) and (5.9):

$$x_D^{\text{diff}} = x_D^{\text{req}} - f_x(L_0/2) = -7.1 \cdot 10^{-15} \text{ m},$$

$$z_D^{\text{diff}} = z_D^{\text{req}} - f_z(L_0/2) = 0 \text{ m}.$$

It is clearly seen that the shape of the cable was successfully controlled.

Equation (4.25) gives the length of the strained cable, which gives

$$L = 100.107 \text{ m}.$$

The elongation of the cable is calculated as the difference between the length of the strained cable and the length of the unstrained cable:

$$\Delta L = L - L_0 = 0.0334 \text{ m}.$$

Example 2.2, III (Continued from Chapter 4)

Calculate H, V_A, s_D and the initial length L_0 of the cable such that

$$z_D^{\text{req}} = f_z(s_D) = 24 \text{ m}. \tag{E.2.2.1}$$

Also calculate the length L of the strained cable and the elongation ΔL of the cable.

The end points of the cable are, respectively, assumed to be located at

$$(x_A, z_A) = (125, 26) \text{ m} \quad \text{and} \quad (x_B, z_B) = (225, 28) \text{ m}.$$

We repeat here the values of three parameters, pertaining to the inextensible cable analysed in the first part of this example:

$$H_{\text{ie}} = 1.7178 \cdot 10^4 \text{ N}, \quad L_{\text{ie}} = 100.246 \text{ m}, \quad s_{D,\text{ie}} = f_{s,\text{ie}}(x_{D,\text{ie}}) = 41.495 \text{ m}.$$

Solution

Since the cable is of asymmetric shape, we solve Equations (5.6) for H, V_A, L_0 and s_D. We then use

$$H_0 = H_{\text{ie}}, \quad L_{0,0} = L_{\text{ie}}, \quad s_{D,0} = s_{D,\text{ie}}, \quad V_{A,0} = \frac{q_c L_{\text{ie}}}{2} = 2005 \text{ N},$$

as starting values for the numerical solution process. The result of the calculation is:

$$H = 1.7174 \cdot 10^4 \text{ N}, \quad V_A = 1659.4 \text{ N}$$

$$L_0 = 100.223 \text{ m}, \quad s_D = 41.486 \text{ m}.$$

Figure E2.2.8 shows the convergence behaviour of the numerical calculation of the above stated parameters (see the end of this chapter).

With the aid of Equations (5.9) and (4.21), it is calculated that

$$z_D^{\text{diff}} = z_D^{\text{req}} - f_z(s_D) = 1.7 \cdot 10^{-13} \text{ m}.$$

We calculate the length of the strained cable according to Equation (4.25), which yields

$$L = 100.246 \text{ m}.$$

The elongation of the cable is computed as

$$\Delta L = L - L_0 = 0.0230 \text{ m}.$$

Example 5.1

In this example, we assume that the bridge of Example 3.1, in Chapter 3, is of Type 1, as shown in Figure 3.3a. When unloaded, the cross sectional area of each main cable is $A_0 = 0.6$ m^2. It is also assumed that the main cables are made of a linearly elastic material with modulus of elasticity $E = 2.05 \cdot 10^{11}$ N/m^2, and density $\rho = 7850$ kg/m^3. Consequently, the axial rigidity of each main cable is $EA_0 = 1.23 \cdot 10^{11}$ N.

The x and z co-ordinates of the ends of Section III of each main cable are, respectively, given by

$$(x_A, z_A) = (1310, 150) \text{ m} \quad \text{and} \quad (x_B, z_B) = (1660, -10) \text{ m}.$$

It is required that the horizontal component of the cable tension is to be equal to

$$H = 2.20998 \cdot 10^8 \text{ N}.$$

This value of H is calculated in Example 3.1, III, in Chapter 7, where the main span section of the main cables is dealt with.

Calculate V_A, L_0, the length L of the strained cable and its elongation ΔL. Plot the cable tension T.

Solution

We calculate the unknown parameters V_A and L_0 by solving the system of equations (5.10). As only two parameters are to be determined, the starting values for the numerical solution procedure are obtained graphically, which gives the starting values $V_{A,0} = 1.1 \cdot 10^8$ N and $L_{0,0} = 384$ m. The result of the calculation is given by

$$V_A = 1.09947 \cdot 10^8 \text{ N}, \qquad L_0 = 384.149 \text{ m}.$$

Figure E3.1.4 shows the convergence behaviour of the numerical solution.

By using Equation (4.25), the length of the strained cable is calculated to give

$$L = 384.908 \text{ m}.$$

Consequently, the elongation of the cable is

$$\Delta L = L - L_0 = 0.759 \text{ m}.$$

Figure E3.1.2 shows the shape of the cable, and Figure E3.1.3 shows the cable tension $T = f_T(s)$ versus $x = f_x(s)$.

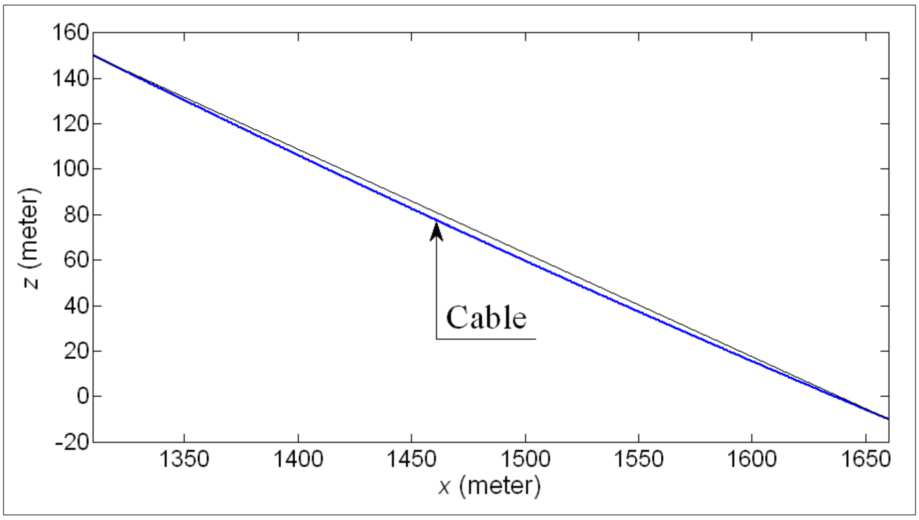


Figure E3.1.2: The shape of the cable. As indicated by the narrow straight line above the cable, the cable is slightly curved due to gravity.

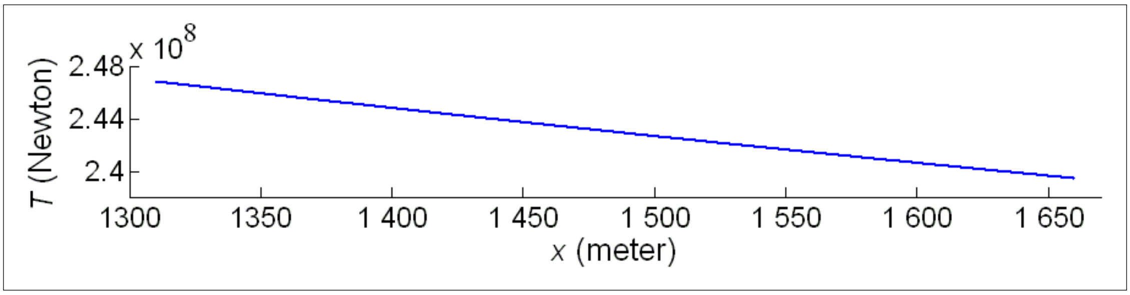


Figure E3.1.3: The co-ordinate $x = f_x(s)$ versus the cable tension $T = f_T(s)$.

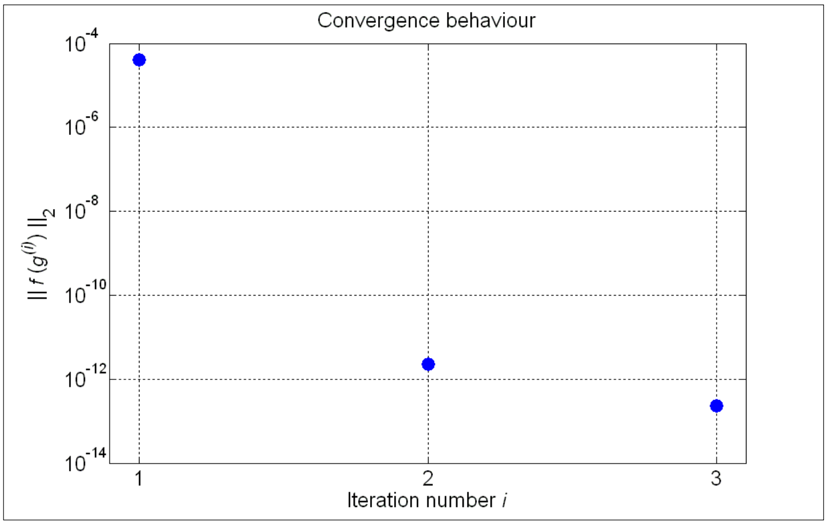


Figure E3.1.4: Convergence behaviour of the numerical calculation of V_A and L_0.

Figure E2.1.3 of Example 2.1, III

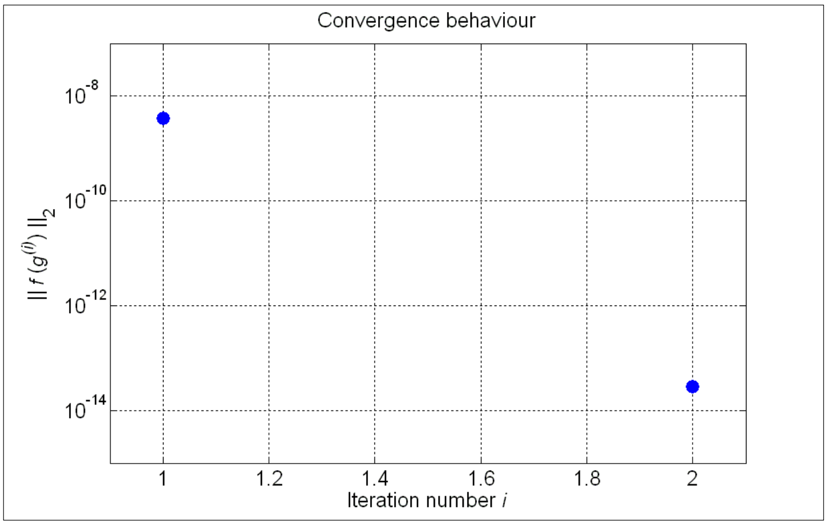


Figure E2.1.3: Convergence behaviour of the numerical calculation of H, V_A and L_0. The number of iterations is 3, but that cannot be seen because $||f(g^{(3)})||_2 = 0$.

Figure E2.2.8 of Example 2.2, III

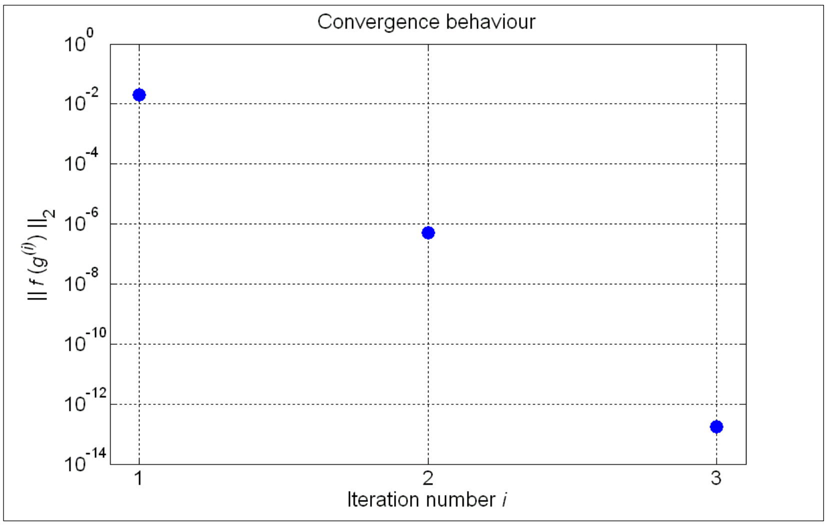


Figure E2.2.8: Convergence behaviour of the numerical calculation of H, V_A L_0 and s_D.

6. Cable loaded by gravity and vertical point forces

As stated in Chapter 3, there are situations where the external loads on cables are approximated as point forces. This applies, for example, to suspension bridges in which the weight of the road deck is transferred to the main cables via vertical cables, so-called hangers. Cables which, in addition to gravity, are assumed to support vertical point forces (hereinafter also called point loads), are in this paper analysed according to a theory presented in Irvine and Sinclair [2]. This theory is an extension of the theory of the linearly elastic catenary described in Chapter 4 of the present paper. Hence, the assumptions made regarding the properties of the cable, and the co-ordinate systems used to describe the shape of the cable, are the same as those of Chapter 4.

We begin by deriving, in detail, the equations for the shape of a cable and the cable tension concerning the case of one point load. The derivation applicable for two point loads is also given in detail, since the final result, Equations (6.19), (6.20) and (6.21), which also include the case of pure gravitational load, can be concluded from these derivations.

For any number of point loads, the shape of the cable is assumed to be given by the functions $x = f_x(s)$, $z = f_z(s)$ and $p = f_p(s)$, which take different forms depending on the number of point loads.

As in Chapters 4 and 5, the functions f_x and f_z may be considered to be functions of several variables, when the values of certain parameters are determined by solving systems of nonlinear equations.

We will also, in the same way as in Chapter 4, make use of the co-ordinate system Ox_cz_c in order to facilitate programming of the equations.

Moreover, the boundary conditions are written on the same form as in Chapter 4, that is

$$x_A = f_x(0) \quad \text{(a)}, \qquad z_A = f_z(0) \quad \text{(b)}, \qquad p_A = f_p(0) = 0 \quad \text{(c)},$$

$$x_B = f_x(L_0) \quad \text{(d)}, \qquad z_B = f_z(L_0) \quad \text{(e)}, \qquad p_B = f_p(L_0) = L \quad \text{(f)}, \tag{6.1}$$

where L is the initially unknown length of the strained cable, (x_A, z_A) and (x_B, z_B) are the position co-ordinates of points A and B, respectively. The arch length co-ordinate s is defined as before.

While the functions $x = f_x(s)$, $z = f_z(s)$ and $p = f_p(s)$ are continuous functions of s, $0 \le s \le L_0$, the derivatives $\dfrac{dx}{ds}$, $\dfrac{dz}{ds}$, $\dfrac{dx}{dp}$ and $\dfrac{dz}{dp}$ are discontinuous at the points of application of the point loads. This means that Equations (4.2), (4.3) and (4.4) are not valid at these points. The discontinuities exist because external loads, other than gravity, are applied as point loads to cables that are assumed to have zero flexural rigidity. Although point forces do not exist in reality, and despite the fact that it is usually more realistic to assume that the slope of the centerline of a cable is continuous at every point, the easily applied theory of Irvine and Sinclair [2] provides realistic results in many problems.

In the solved shape control problems of the examples in Chapters 7, where point forces and cables of large diameter are involved, the solutions of Irvine and Sinclair [2] give results with small jumps of the value of the slope $\dfrac{dz}{dx} = \dfrac{dz}{ds} \cdot \dfrac{ds}{dx}$ of the cable centerline. This means that it is likely that the assumption of negligible flexural rigidity of the cable is realistic, and we therefore assume that the solutions given in Chapter 7 are practically useful.

6.1 One vertical point force

We now assume that the arc-length co-ordinate of the point of application of the force $\boldsymbol{F}_1$ is s_1, $0 < s_1 < L_0$, in the unstrained configuration, whereas this point has the initially unknown co-ordinates $p_1 = f_p(s_1)$ and $(x_1, z_1) = (f_x(s_1), f_z(s_1))$ in the strained configuration.

It is clear that in the interval $0 \leq s < s_1$, equilibrium Equations (4.5) and (4.6) are also valid for the cable shown in Figure 6.1. Therefore, the functions $f_x(s)$, $f_z(s)$, $f_T(s)$ and $f_p(s)$ are, in this interval of s, the same as those derived in Chapter 4.

Horizontal and vertical equilibrium of the portion of the strained cable shown in Figure 6.1 requires for the interval $s_1 < s \leq L_0$, that

$$T\frac{dx}{dp} = H, \quad s_1 < s \leq L_0, \tag{6.2}$$

$$T\frac{dz}{dp} = q_c s - V_A + F_1, \quad s_1 < s \leq L_0, \tag{6.3}$$

where F_1 is the magnitude of $\boldsymbol{F}_1$, V_A is the scalar component of the vertical reaction force at point A, H is the horizontal component of the cable tension $T = f_T(s)$, and $q_c s$ is the self weight of the shown portion of the cable.

While the equation for the cable tension, applicable for $0 \leq s < s_1$, is the same as Equation (4.10), the expression for the cable tension regarding the interval $s_1 < s \leq L_0$, is obtained by inserting dx/dp and dz/dp according to Equations (6.2) and (6.3), into Equation (4.9), which is then solved for T. The equations for the cable tension can then be written as

$$T = f_T(s) = \sqrt{H^2 + (q_c s - V_A)^2}, \quad 0 \leq s < s_1, \tag{6.4a}$$

$$T = f_T(s) = \sqrt{H^2 + (q_c s - V_A + F_1)^2}, \quad s_1 < s \leq L_0. \tag{6.4b}$$

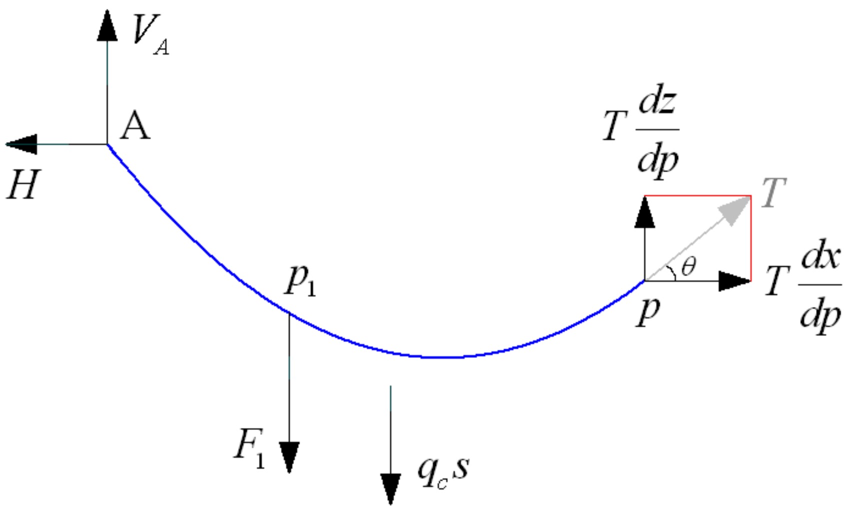


Figure 6.1: Forces acting on a portion of the strained cable. The length of this portion is $p = f_p(s)$, and its self weight is $q_c s$.

The geometry relation, given by Equation (4.9), is not valid at $s = s_1$ since the slope of the center line of the cable is assumed to be discontinuous at p_1. Therefore, Equations (6.4) are not valid at $s = s_1$ either.

The functions $x = f_x(s)$ and $z = f_z(s)$, applicable for $0 \leq s \leq s_1$, are given by Equations (4.16) and (4.21), respectively. In order to derive the function $x = f_x(s)$, valid in the interval $s_1 \leq s \leq L_0$, we insert Equation (6.4b) into Equations (6.2), which results in the relation

$$\frac{dx}{dp} = \frac{H}{T} = \frac{H}{\sqrt{H^2 + (q_c s - V_A + F_1)^2}}. \tag{6.5}$$

Insertion of T given by Equation (6.4b), into Equation (4.3) gives

$$\frac{dp}{ds} = \frac{T}{EA_0} + 1 = \frac{1}{EA_0}\sqrt{H^2 + (q_c s - V_A + F_1)^2} + 1. \tag{6.6}$$

Equations (6.5) and (6.6) are inserted into Equation (4.11) which, upon integration, yields

$$\int dx = \int \frac{H}{EA_0} ds + \int \frac{H}{\sqrt{H^2 + (q_c s - V_A + F_1)^2}}\, ds \quad \Leftrightarrow \tag{6.7}$$

$$x = f_x(s) = \frac{H}{EA_0} s + \frac{H}{q_c} \sinh^{-1}\left(\frac{q_c s - V_A + F_1}{H}\right) + C, \tag{6.8}$$

where C is a constant of integration. Equation (6.8) is to be valid in the interval $s_1 \leq s \leq L_0$, and we require of the function $f_x(s)$ that $x_1 = f_x(s_1)$. In order to ensure that the function $f_x(s)$ is continuous, we assume that x_1 is given by Equation (4.16) at $s = s_1$. Then, by inserting $s = s_1$ and $x = x_1$ into Equation (6.8), we get an equation that is solved for C. The equations for x can now be written as

$$\begin{aligned} x = f_x(s) &= x_A + \frac{H}{EA_0} s + \\ &+ \frac{H}{q_c}\left(\sinh^{-1}\left(\frac{q_c s - V_A}{H}\right) - \sinh^{-1}\left(-\frac{V_A}{H}\right)\right), \quad 0 \leq s \leq s_1, \end{aligned} \tag{6.9a}$$

$$\begin{aligned} x = f_x(s) &= x_A + \frac{H}{EA_0} s + \\ &+ \frac{H}{q_c}\left(\sinh^{-1}\left(\frac{q_c s - V_A + F_1}{H}\right) - \sinh^{-1}\left(\frac{q_c s_1 - V_A + F_1}{H}\right)\right) + \\ &+ \frac{H}{q_c}\left(\sinh^{-1}\left(\frac{q_c s_1 - V_A}{H}\right) - \sinh^{-1}\left(-\frac{V_A}{H}\right)\right), \quad s_1 \leq s \leq L_0. \end{aligned} \tag{6.9b}$$

We shall now derive the equations for the position co-ordinate $z = f_z(s)$. Equation (6.3) together with Equation (6.4b) implies that

$$\frac{dz}{dp} = \frac{1}{T}\left(q_c s - V_A + F_1\right) = \frac{q_c s - V_A + F_1}{\sqrt{H^2 + \left(q_c s - V_A + F_1\right)^2}}. \tag{6.10}$$

Insertion of Equations (6.10) and (6.6) into to Equation (4.17), followed by integration, results in

$$\int dz = \int \frac{1}{EA_0}\left(q_c s - V_A + F_1\right) ds + \int \frac{q_c s - V_A + F_1}{\sqrt{H^2 + \left(q_c s - V_A + F_1\right)^2}} ds \quad \Leftrightarrow \tag{6.11}$$

$$z = f_z(s) = \frac{1}{EA_0}\left(\frac{q_c}{2}s^2 - V_A s + F_1 s\right) + \frac{1}{q_c}\sqrt{H^2 + \left(q_c s - V_A + F_1\right)^2} + C, \tag{6.12}$$

where C is a constant of integration. Equation (6.12) is to be valid in the interval $s_1 \leq s \leq L_0$, and it is required that $z_1 = f_z(s_1)$. In order to ensure that the function $f_z(s)$ is continuous, we assume that z_1 is given by Equation (4.21) at $s = s_1$. We calculate the constant C by solving an equation that is created by inserting $s = s_1$ and $z = z_1$ into Equation (6.12). The equations for z are as follows:

$$\begin{aligned} z = f_z(s) = z_A &+ \frac{1}{EA_0}\left(\frac{q_c}{2}s^2 - V_A s\right) + \\ &+ \frac{1}{q_c}\left(\sqrt{H^2 + \left(q_c s - V_A\right)^2} - \sqrt{H^2 + {V_A}^2}\right), \quad 0 \leq s \leq s_1, \end{aligned} \tag{6.13a}$$

$$\begin{aligned} z = f_z(s) = z_A &+ \frac{1}{EA_0}\left(\frac{q_c}{2}s^2 - V_A s + F_1 s - F_1 s_1\right) + \\ &+ \frac{1}{q_c}\left(\sqrt{H^2 + \left(q_c s - V_A + F_1\right)^2} - \sqrt{H^2 + \left(q_c s_1 - V_A + F_1\right)^2}\right) + \\ &+ \frac{1}{q_c}\left(\sqrt{H^2 + \left(q_c s_1 - V_A\right)^2} - \sqrt{H^2 + {V_A}^2}\right), \quad s_1 \leq s \leq L_0. \end{aligned} \tag{6.13b}$$

If the values of E, A_0, q_c and L_0 are known, then there are two unknown constants, H and V_A, that must be determined before the solution is complete. By inserting $s = L_0$ and $x = x_B$ into Equation (6.9b), and inserting $s = L_0$ and $z = z_B$ into Equation (6.12b), we get a nonlinear system of equations which is solved numerically for H and V_A.

6.2 Two vertical point forces

It is assumed that the arc-length co-ordinates of the points of application of the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are s_1 and s_2, respectively, in the unstrained configuration, whereas these points, respectively, have the initially unknown co-ordinates $p_1 = f_p(s_1)$, $(x_1, z_1) = (f_x(s_1), f_z(s_1))$, $p_2 = f_p(s_2)$, and $(x_2, z_2) = (f_x(s_2), f_z(s_2))$, in the strained configuration. It holds that $0 < s_1 < s_2$ and $s_1 < s_2 < L_0$.

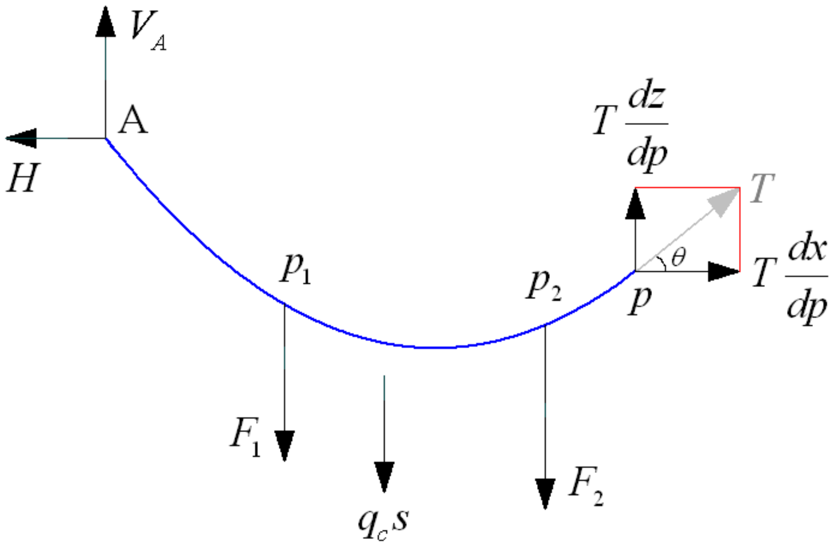


Figure 6.2: Forces acting on a portion of the strained cable that is loaded by gravity and two point forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$.

In the interval $0 \leq s < s_1$, the equilibrium Equations (4.5) and (4.6) are valid, and in the interval $s_1 < s < s_2$, we have that equilibrium Equations (6.2) and (6.3) holds. This means that in the interval $0 \leq s \leq s_2$, the equations for x and z are given by Equations (6.9) and (6.13), respectively, whereas the equations for T are given by Equations (6.4).

Horizontal and vertical equilibrium of the portion of the strained cable shown in Figure 6.2 requires for the interval $s_2 < s \leq L_0$ that

$$T\frac{dx}{dp} = H, \quad s_2 < s \leq L_0, \tag{6.14}$$

$$T\frac{dz}{dp} = q_c s - V_A + F_1 + F_2, \quad s_2 < s \leq L_0, \tag{6.15}$$

where F_2 is the magnitude of $\boldsymbol{F_2}$, and $q_c s$ is the self weight of the shown portion of the cable.

We obtain the expression for the cable tension, regarding the interval $s_2 < s \leq L_0$, by inserting dx/dp and dz/dp, as given by Equations (6.14) and (6.15), into Equation (4.9), which is then solved for the tension T. The equations for T become

$$T = f_T(s) = \sqrt{H^2 + (q_c s - V_A)^2}\ , \quad 0 \leq s < s_1, \tag{6.16a}$$

$$T = f_T(s) = \sqrt{H^2 + (q_c s - V_A + F_1)^2}, \quad s_1 < s < s_2, \tag{6.16b}$$

$$T = f_T(s) = \sqrt{H^2 + (q_c s - V_A + F_1 + F_2)^2}, \quad s_2 < s \leq L_0. \tag{6.16c}$$

The geometry relation, Equation (4.9), is not valid at $s = s_1$ and $s = s_2$ since the slope of the center line of the cable is assumed to be discontinuous at p_1 and p_2. Therefore, Equations (6.16) are not valid at $s = s_1$ and $s = s_2$.

By assuming that x_2 and z_2, respectively, are given by Equations (6.9b) and (6.13b) at $s = s_2$, and then proceeding in the same way as we did in Section 6.1, we obtain the equations for x and z, which can be written as

$$x = f_x(s) = x_A + \frac{H}{EA_0}s + \\ + \frac{H}{q_c}\left(\sinh^{-1}\left(\frac{q_c s - V_A}{H}\right) - \sinh^{-1}\left(-\frac{V_A}{H}\right)\right), \quad 0 \leq s \leq s_1, \tag{6.17a}$$

$$x = f_x(s) = x_A + \frac{H}{EA_0}s + \\ + \frac{H}{q_c}\left[\sinh^{-1}\left(\frac{q_c s - V_A + F_1}{H}\right) - \sinh^{-1}\left(\frac{q_c s_1 - V_A + F_1}{H}\right) + \\ + \sinh^{-1}\left(\frac{q_c s_1 - V_A}{H}\right) - \sinh^{-1}\left(-\frac{V_A}{H}\right)\right], \quad s_1 \leq s \leq s_2, \tag{6.17b}$$

$$x = f_x(s) = x_A + \frac{H}{EA_0}s + \\ + \frac{H}{q_c}\left[\sinh^{-1}\left(\frac{q_c s - V_A + F_1 + F_2}{H}\right) - \sinh^{-1}\left(\frac{q_c s_2 - V_A + F_1 + F_2}{H}\right) + \\ + \sinh^{-1}\left(\frac{q_c s_2 - V_A + F_1}{H}\right) - \sinh^{-1}\left(\frac{q_c s_1 - V_A + F_1}{H}\right) + \\ + \sinh^{-1}\left(\frac{q_c s_1 - V_A}{H}\right) - \sinh^{-1}\left(-\frac{V_A}{H}\right)\right], \quad s_2 \leq s \leq L_0, \tag{6.17c}$$

$$z = f_z(s) = z_A + \frac{1}{EA_0}\left(\frac{q_c}{2}s^2 - V_A s\right) + \\ + \frac{1}{q_c}\left(\sqrt{H^2 + (q_c s - V_A)^2} - \sqrt{H^2 + {V_A}^2}\right), \quad 0 \leq s \leq s_1, \tag{6.18a}$$

$$z = f_z(s) = z_A + \frac{1}{EA_0}\left(\frac{q_c}{2}s^2 - V_A s + F_1 s - F_1 s_1\right) + \\ + \frac{1}{q_c}\left(\sqrt{H^2 + (q_c s - V_A + F_1)^2} - \sqrt{H^2 + (q_c s_1 - V_A + F_1)^2} + \\ + \sqrt{H^2 + (q_c s_1 - V_A)^2} - \sqrt{H^2 + {V_A}^2}\right), \quad s_1 \leq s \leq s_2, \tag{6.18b}$$

$$
\begin{aligned}
z = f_z(s) = z_A + \frac{1}{EA_0}\left(\frac{q_c}{2}s^2 - V_A s + (F_1+F_2)s - F_1 s_1 - F_2 s_2\right) + \\
+ \frac{1}{q_c}\Bigg(\sqrt{H^2 + (q_c s - V_A + F_1 + F_2)^2} - \sqrt{H^2 + (q_c s_2 - V_A + F_1 + F_2)^2} + \\
+ \sqrt{H^2 + (q_c s_2 - V_A + F_1)^2} - \sqrt{H^2 + (q_c s_1 - V_A + F_1)^2} + \\
+ \sqrt{H^2 + (q_c s_1 - V_A)^2} - \sqrt{H^2 + {V_A}^2}\Bigg), \quad s_2 \le s \le L_0.
\end{aligned} \tag{6.18c}
$$

If the values of E, A_0, q_c and L_0 are known, there are two unknown constants, H and V_A, that must be determined before the solution is complete. By inserting $s = L_0$ and $x = x_B$ into Equation (6.17c), and inserting $s = L_0$ and $z = z_B$ into Equation (6.18c), we get a nonlinear system of equations which is solved numerically for H and V_A.

6.3 The general case

The equations for the functions $T = f_T(s)$, $x = f_x(s)$ and $z = f_z(s)$ will grow immensely in size as the number of included point loads increases. However, by studying Equations (6.16), (6.17), (6.18) and the derivations of these equations, it can be seen that successive inclusion of additional point loads follows a certain scheme. This means that the equations for $f_T(s)$, $f_x(s)$ and $f_z(s)$, valid both for the case of pure gravitational load, as well as for the case of many point loads, can be written in compact form. To this end, we assume that $N_F \ge 0$ is the number of vertical point loads that act on the cable. It is also assumed that the arc-length co-ordinate associated with the point of application of the point load $\boldsymbol{F}_j$, of magnitude F_j, $j = 1, 2, \ldots, N_F$, is s_j in the unstrained configuration, whereas this point has arc-length co-ordinate $p_j = f_p(s_j)$, and Cartesian co-ordinates $(x_j, z_j) = (f_x(s_j), f_z(s_j))$, in the strained configuration. It holds that

$$
0 < s_1 < \cdots < s_j < s_{j+1} < \cdots < s_{N_F} < L_0.
$$

Although no point load is associated with the co-ordinate $s = L_0$, it turns out to be convenient to define that $s_{N_F+1} = L_0$.

In order to make the general equations for $f_T(s)$, $f_x(s)$ and $f_z(s)$ able to handle both the case of pure gravitational load, as well as the interval $0 \le s < s_1$ if $N_F \ge 1$, we create two additional point loads of magnitude $F_{-1} = F_0 = 0$, which, respectively, are assumed to be associated with the arc-length co-ordinates $s_{-1} = s_0 = 0$, in the unstrained configuration.

The general equations for $T = f_T(s)$ can then be written as

$$
T = f_T(s) = \sqrt{H^2 + \left(q_c s - V_A + \sum_{j=0}^{n} F_j\right)^2}\,, \tag{6.19}
$$

where n is determined as a function of s and N_F according to:

If $N_F = 0$: $n = 0, \quad 0 \le s \le L_0$

$$\text{If } N_F \ge 1\text{:} \quad \begin{cases} n = 0 \quad \text{if} \quad s_0 \le s < s_1 \\ n = j \quad \text{if} \quad s_j < s < s_{j+1}, \ 1 \le j \le (N_F - 1) \\ n = N_F \quad \text{if} \quad s_{N_F} < s \le L_0 \end{cases}$$

By taking advantage of the local co-ordinate system Ox_cz_c, shown in Figure 4.3, and Equations (4.31) and (4.32), the general equations for $x = f_x(s)$ and $z = f_z(s)$ may, respectively, be given by the following expressions:

$$x = f_x(s) = x_A + f_{x_c}(s) = x_A + \frac{H}{EA_0}s +$$

$$+ \frac{H}{q_c}\left[\sinh^{-1}\left(\frac{1}{H}\left(q_c s - V_A + \sum_{j=0}^{n} F_j\right)\right) - \sinh^{-1}\left(\frac{1}{H}\left(q_c s_n - V_A + \sum_{j=0}^{n} F_j\right)\right) +\right.$$

$$\left. + \sum_{k=0}^{n}\left\{\sinh^{-1}\left(\frac{1}{H}\left(q_c s_k - V_A + \sum_{j=-1}^{k-1} F_j\right)\right) - \sinh^{-1}\left(\frac{1}{H}\left(q_c s_{k-1} - V_A + \sum_{j=-1}^{k-1} F_j\right)\right)\right\}\right], \tag{6.20}$$

$$z = f_z(s) = z_A + f_{z_c}(s) =$$

$$= z_A + \frac{1}{EA_0}\left(\left(\frac{q_c}{2}\right)s^2 - V_A s + s\sum_{j=0}^{n} F_j - \sum_{j=0}^{n} F_j s_j\right) +$$

$$+ \frac{1}{q_c}\left[\sqrt{H^2 + \left(q_c s - V_A + \sum_{j=0}^{n} F_j\right)^2} - \sqrt{H^2 + \left(q_c s_n - V_A + \sum_{j=0}^{n} F_j\right)^2} +\right.$$

$$\left. + \sum_{k=0}^{n}\left(\sqrt{H^2 + \left(q_c s_k - V_A + \sum_{j=-1}^{k-1} F_j\right)^2} - \sqrt{H^2 + \left(q_c s_{k-1} - V_A + \sum_{j=-1}^{k-1} F_j\right)^2}\right)\right], \tag{6.21}$$

where, in Equations (6.20) and (6.21), it holds that

$$N_F \ge 0, \quad s_{-1} = s_0 = F_{-1} = F_0 = 0.$$

It is the value of s that decides the value of n in Equations (6.20) and (6.21). Consequently, n must be determined for every s that is to be inserted into these equations. Since both x and z

are continuous functions of $s,$ it is, in principle, possible to let n be constant and equal to j for every s in the interval $s_j \leq s \leq s_{j+1},$ as is the case in, for example, Equations (6.17) and (6.18). However, this method for determining n as a function of s is not convenient for the shape control problems dealt with in Chapter 7. For reasons described in Chapter 7, we, instead, let n be constant and equal to j for every s in the interval $s_j \leq s < s_{j+1},$ $j = 0, 1, 2, \ \ldots \ , N_F$, except for $s = s_{N_F+1} = L_0$, in which case we set $n = N_F$. For example, if $N_F = 0$, then $n = 0,$ or if $s_2 \leq s < s_3$, $N_F \geq 3$, we get $n = 2$ (see Figure 6.3).

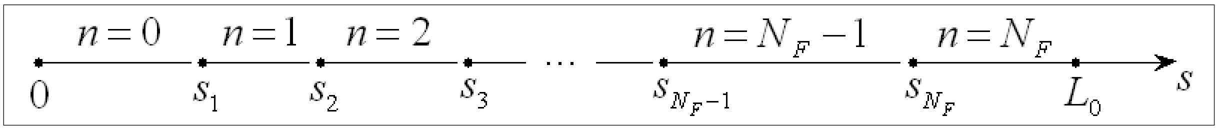


Figure 6.3: The figure illustrates how n is determined as a function of s. For instance, n is given the value $n = 1$ for every s in the interval $s_1 \leq s < s_2,$ or n is given the value $n = N_F$ for every s in the interval $s_{N_F} \leq s \leq L_0$. The fact that, for instance, $s = s_2$ is not included in the interval $s_1 \leq s < s_2,$ is shown as a gap in the s-line.

If the values of $E,$ $A_0,$ $q_c,$ L_0 and every $s_j, \ j = 1, 2, \ \ldots \ , N_F$, are known, the remaining parameters H and V_A must be determined before the solution is complete. By inserting $s = L_0$ and $x = x_B$ into Equation (6.20), and inserting $s = L_0$ and $z = z_B$ into Equation (6.21), we get a system of nonlinear equations which is written as

$$0 = -x_{cB} + f_{x_c}(H, V_A)|_{s=L_0} \tag{6.22a}$$

$$0 = -z_{cB} + f_{z_c}(H, V_A)|_{s=L_0}, \tag{6.22b}$$

where

$$x_{cB} = x_B - x_A, \quad z_{cB} = z_B - z_A.$$

This system of equations is solved numerically for H and V_A. The starting values H_0 and $V_{A,0}$, necessary for the numerical solution procedure, can be obtained by plotting the plane $f_0(H, V_A) = 0,$ together with the graphs of the two functions

$$f_1\,(H, V_A) = -x_{cB} + f_{x_c}(H, V_A)|_{s=L_0} \tag{6.23a}$$

$$f_2\,(H, V_A) = -z_{cB} + f_{z_c}(H, V_A)|_{s=L_0}. \tag{6.23b}$$

While the value of V_A may be either positive or negative, depending on the shape of the cable, the value of H is always positive since the cable only sustains tensile stresses.

If the supports at the ends of the cable are on the same vertical level, and the external vertical loading is symmetric with respect to the cable midpoint, then the shape of the cable is symmetric and, consequently, the scalar component of the reaction force $\boldsymbol{V_A}$ is calculated as

$$V_A = V_A^{\mathrm{sym}} = \frac{1}{2}\left(q_c L_0 + \sum_{j=0}^{N_F} F_j\right). \tag{6.24}$$

It can be shown that $V_A = V_A^{\text{sym}}$ is a solution to Equation (6.22b) if $z_A = z_B$, and that this solution is independent of H. We therefore determine the value of the horizontal component of the cable tension by solving the following equation for H:

$$0 = -x_{cB} + f_{x_c}(H)|_{s=L_0,\ V_A=V_A^{\text{sym}}}. \tag{6.25}$$

It is possible to calculate the arc-length co-ordinate $p = f_p(s)$ according to an analytical expression, see Irvine and Sinclair [2], but in the present paper, p is calculated numerically. In order to determine p, we start with the following expressions:

$$p = f_p(s) = \int_0^p dp^* = \int_0^s \sqrt{\left(\frac{dx}{ds^*}\right)^2 + \left(\frac{dz}{ds^*}\right)^2} ds^*, \tag{6.26}$$

where $p^* = p$ and $s^* = s$ are integration variables,

$$\frac{dx}{ds} = \frac{df_x}{ds}(s) = \frac{H}{EA_0} + \left(1 + \frac{1}{H^2}\left(q_c s - V_A + \sum_{j=0}^{n} F_j\right)^2\right)^{-\frac{1}{2}} \tag{6.27}$$

and

$$\frac{dz}{ds} = \frac{df_z}{ds}(s) = \frac{1}{EA_0}\left(q_c s - V_A + \sum_{j=0}^{n} F_j\right) + \frac{q_c s - V_A + \sum_{j=0}^{n} F_j}{\sqrt{H^2 + \left(q_c s - V_A + \sum_{j=0}^{n} F_j\right)^2}}. \tag{6.28}$$

The arc-length co-ordinate $p = f_p(s)$ is a continuous function of s, $0 \le s \le L_0$, but for $N_F \ge 1$, both df_x/ds and df_z/ds are discontinuous functions of s, and, furthermore, they are not defined at any $s = s_j,\ j = 1, 2,\ \ldots\ , N_F$. In the present paper, the integral (6.26) is evaluated numerically. In doing so, we use the fact that for a fixed n, the functions df_x/ds and df_z/ds are defined mathematically on the interval $s_j \le s \le s_{j+1}$, and, therefore, the value of p is determined by adding the numerically calculated lengths of a number of line segments. This method can schematically be written as

$$p = f_p(s) = \int_0^p dp^* = \int_{p_0}^{p_1} dp^* + \int_{p_1}^{p_2} dp^* + \int_{p_2}^{p_3} dp^* + \ \cdots\ + \int_{p_{n_s}}^{p} dp^*, \tag{6.29}$$

where $p_0 = 0$,

$$\int_{p_a}^{p_{a+1}} dp^* = \int_{s_a}^{s_{a+1}} \sqrt{\left(\frac{dx}{ds^*}\right)^2 + \left(\frac{dz}{ds^*}\right)^2} ds^*\Bigg|_{n=a}, \tag{6.30a}$$

$$\int_{p_{n_s}}^{p} dp^* = \int_{s_{n_s}}^{s} \sqrt{\left(\frac{dx}{ds^*}\right)^2 + \left(\frac{dz}{ds^*}\right)^2} ds^*\Bigg|_{n=n_s}, \tag{6.30b}$$

$$a = 0,\ 1,\ 2,\ \dots\ ,\ n_s - 1,$$

and n_s, is the particular value of n that is decided by the inserted value of s in Equation (6.29). We determine the value of n_s as a function of s according to the rule illustrated in Figure 6.3. In Equations (6.30), the notations $|_{n=a}$ and $|_{n=n_s}$ indicate that n is kept constant and equal to a, or equal to n_s, in Equations (6.27) and (6.28), when the integrals are calculated. By using Equations (6.27) to (6.30), the length of the strained cable can be calculated as

$$L = \int_0^L dp = \int_0^{L_0} \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2} ds. \tag{6.31}$$

Calculation of the derivative dz/dx is done numerically in the present paper according to

$$\frac{dz}{dx} = \frac{dz}{ds}\frac{ds}{dx}, \tag{6.32}$$

where dz/ds and dx/ds are given by Equations (6.27) and (6.28), respectively. The derivative dz/dx is a discontinuous function of s if $N_F \geq 1$.

Solution of the system of Equations (6.22), as well as Equation (6.25), is performed using Newton's method, see Appendix A for details.

The examples of the present chapter concern cables of symmetric shape, which means that Equation (6.25) is solved for H. The single entry of the Jacobian matrix, given by Equation (A.3), is then calculated numerically according to Equation (A.4) with $h = 0.1$.

Example 2.1, IV (Continued from Chapter 5)

In addition to gravity, a downwardly directed vertical point force of magnitude $F_1 = 20 \text{ kN}$ is applied to the cable at midspan. The initial length of the cable is taken to be $L_0 = 100.073 \text{ m}$, which is the length that was previously calculated to give the cable a sag of $d_v = 2 \text{ m}$, when it hangs under the action of gravity only.

Determine H, V_A, $z_D = f_z(L_0/2)$, the length L of the strained cable and the elongation ΔL of the cable. Plot the tension $T = f_T(s)$ versus the co-ordinate $x = f_x(s)$.

Solution

The shape of the cable is symmetric since the ends of the cable are at the same vertical level, and because the force $\boldsymbol{F}_1$ is applied at midspan. Therefore, we calculate the scalar component of the reaction force $\boldsymbol{V_A}$ as

$$V_A = \frac{q_c L_0 + F_1}{2} = 1.20015 \cdot 10^4 \text{ N}.$$

With V_A known, Equation (6.25) is solved numerically, which gives

$$H = 1.49075 \cdot 10^5 \text{ N}.$$

The starting value for the numerical solution procedure $H_0 = 1.5 \cdot 10^5 \text{ N}$, was obtained graphically. Figure E2.1.4 shows the shape of the cable, whereas the convergence behaviour of the numerical calculation of H is shown in Figure E2.1.6.

According to Equation (6.21), the z co-ordinate of the point of application of the point force is

$$z_D = f_z(L_0/2) = -3.690 \text{ m}.$$

The length of the strained cable is computed by using Equation (6.31), which yields

$$L = 100.273 \text{ m}.$$

This means that the cable has lengthen

$$\Delta L = L - L_0 = 0.199 \text{ m}.$$

The cable tension is plotted in Figure E2.1.5.

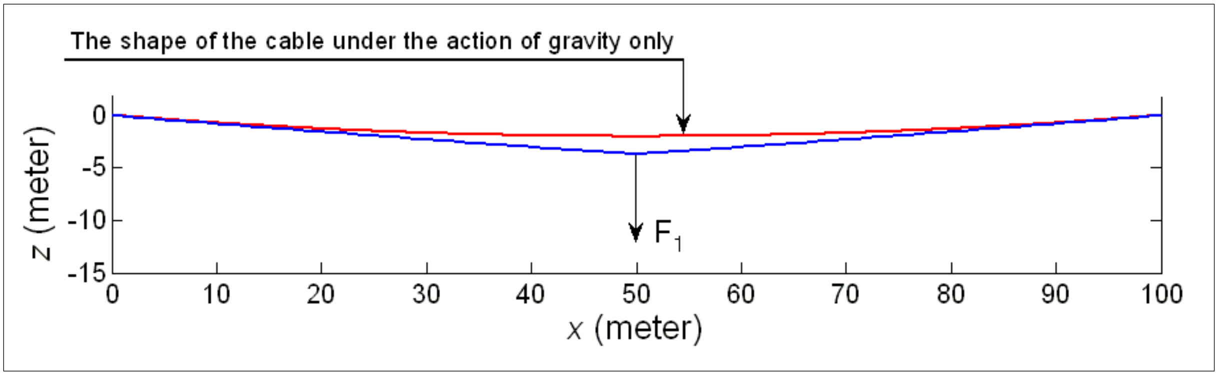


Figure E2.1.4: The shape of the cable under the action of gravity only, and the shape of the cable under the action of both gravity and the vertical external point force $\boldsymbol{F}_1$.

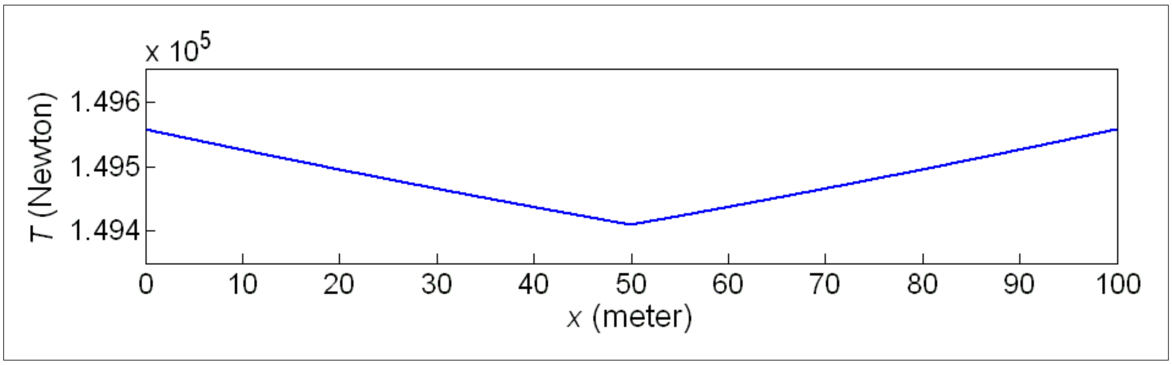


Figure E2.1.5: The co-ordinate $x = f_x(s)$ versus the cable tension $T = f_T(s)$.

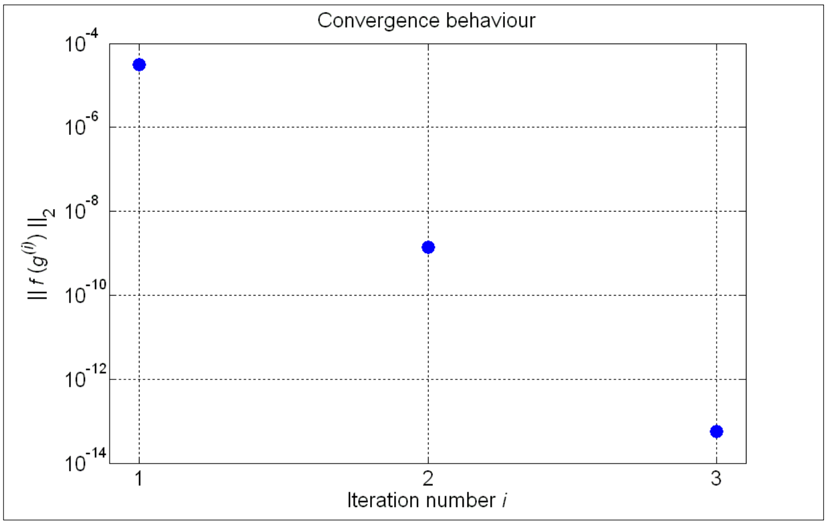


Figure E2.1.6: Convergence behaviour of the numerical calculation of H.

Example 3.1, II (continued from Chapter 3)

We now include axial elasticity in the analysis according to the theory described in this chapter. The external loads on the main cables, other than gravity, are applied as vertical point forces representing the forces from the hangers.

As stated in Part I of this example, the main span section of each main cable has a span of $d_h = 1300$ m, and is expected to have a sag of $d_v = 150$ m, when no vehicles are on the bridge.

When unloaded, the cross sectional area of each main cable is $A_0 = 0.6$ m^2. We assume that the cable material is linearly elastic with modulus of elasticity $E = 2.05 \cdot 10^{11}$ N/m^2, and density $\rho = 7850$ kg/m^3. Consequently, the axial rigidity of each main cable is $EA_0 = 1.23 \cdot 10^{11}$ N.

The road deck is horizontal, and its constant weight per unit length along the main span is $q_{L,\text{total}} = 2.2 \cdot 10^5$ N/m. Each main cable carries half the weight of the road deck, which means that

$$q_L = \frac{q_{L,\text{total}}}{2} = 1.1 \cdot 10^5 \text{ N/m}.$$

The weight of the main span section of the bridge deck is assumed to be transferred to each main cable via $N_F = 80,$ vertical and equally spaced hangers of negligible weight, along the main span of the bridge.

It is also assumed that the towers are rigid and that the main cables cannot move relative to the towers.

The subsequent analysis concerns the main span section of each main cable, which is assumed to carry half of the portion of the bridge deck that stretches horizontally from $x = x_A$ to $x = x_B$.

Assuming that the length of the unstrained cable is $L_0 = L_{\text{ie}}$, calculate H, V_A, the length L of the strained cable, the elongation ΔL of the cable, the maximum cable tension and the z co-ordinate of the lowest point of the cable centerline. Also calculate the x co-ordinates of the points of application of the point loads.

For convenience, the values of some parameters given in Section (b) of Part I of this example are repeated here:

$$(x_A, z_A) = (0, 150) \text{ m}, \quad (x_B, z_B) = (1300, 150) \text{ m},$$

$$H_{\text{ie}} = 2.200 \cdot 10^8 \text{ N}, \quad L_{\text{ie}} = 1344.781 \text{ m}, \quad T_{A,\text{ie}} = T_{B,\text{ie}} = 2.423 \cdot 10^8 \text{ N}.$$

Solution

Since the hangers are assumed to be equally spaced, we calculate the horizontal distance between the centerlines of two adjacent hangers as

$$\Delta x = \frac{x_B - x_A}{N_F} = \frac{1300}{80} = 16.25 \text{ m}.$$

It is assumed that each hanger carries an equally large portion of the bridge deck, and that the weight of such a portion is $q_L \Delta x$. Consequently, the magnitude of the tensile forces $\boldsymbol{F}_j$, $j = 1, 2, \ \dots\ , N_F$, sustained by each hanger, are calculated as

$$F_j = q_L \Delta x = 1.788 \cdot 10^6 \text{ N}, \quad j = 1, 2, \; \dots \; N_F, \tag{E3.1.1}$$

The total weight of the main span section of each main cable is given by

$$W_{\text{ie}} = q_c L_{\text{ie}} = 6.214 \cdot 10^7 \text{ N}, \tag{E3.1.2}$$

where the self-weight per unit length of each main cable is calculated as

$$q_c = A_0 \rho g = 4.621 \cdot 10^4 \text{ N/m}. \tag{E3.1.3}$$

The point loads $\boldsymbol{F}_j$, $j = 1, 2, \; \dots \;, N_F$, that act on the main cables are given by Equation (E3.1.1), and each point load is associated with a value s_i of the arc-length co-ordinate s. Every s_i is calculated according to Equation (3.16) at

$$x_j = x_A + \frac{\Delta x}{2} + \Delta x (j - 1), \quad j = 1, 2, \; \dots \;, N_F. \tag{E3.1.4}$$

The main cables assume a symmetric shape under the action of the external load and, therefore, the scalar component of the vertical reaction force at point A is calculated as

$$V_A = \frac{1}{2} \left(W_{\text{ie}} + \sum_{j=0}^{80} F_j \right) = 1.026 \cdot 10^8 \text{ N}. \tag{E3.1.5}$$

With $H_0 = H_{\text{ie}}$ as the starting value, we solve Equation (6.25) numerically for H to give

$$H = 2.152 \cdot 10^8 \text{ N}.$$

Figure E3.1.2 shows the convergence behaviour of the solution process.

Now that both H and V_A are known, we can use Equation (6.31) to calculate the length of the strained cable:

$$L = 1347.222 \text{ m}.$$

The cable has, as a result of the loading by gravity and the external forces, lengthen

$$\Delta L = L - L_0 = 2.441 \text{ m}.$$

The maximum tension occurs at both ends of the cable where, according to Equation (6.19), we get

$$T_{\text{max}} = f_T(0) = f_T(L_0) = 2.384 \cdot 10^8 \text{ N}.$$

Equation (6.21) is used to compute the z co-ordinate of the lowest point of the cable centerline, which gives

$$z_D = f_z(L_0/2) = -3.972 \text{ m}.$$

Calculating the difference between the co-ordinate z_D of the elastic cable, and the co-ordinate $z_{D,\text{ie}} = 0$ m, of the inextensible cable, yields

$$z_D - z_{D,\text{ie}} = -3.972 \text{ m.}$$

Moreover, computation of

$$||x_j^{\text{diff}}||_2 = ||x_j - f_x(s_j)||_2, \quad j = 1, 2, \ \ldots \ , N_F,$$

where x_j is given by Equation (E3.1.4), gives results that are significantly different from zero. At most, the difference is

$$\max(||x_j^{\text{diff}}||_2) = 0.458 \text{ m}, \ j = 1, 2, \ \ldots \ , N_F.$$

The calculation of the magnitudes of the point loads F_j, $j = 1, 2, \ \ldots \ N_F$, according to Equation (E3.1.1), relies on the assumption that the hangers are vertical, and that the co-ordinate x_j, $j = 1, 2, \ \ldots \ N_F$, of each hanger is prescribed. If the co-ordinates x_j, $j = 1, 2, \ \ldots \ N_F$, deviate significantly from the prescribed values, it is likely that the magnitudes of the loads, applied to the main cables by the hangers, deviate significantly from those given by Equation (E3.1.1). In the present example, the hangers are assumed to be vertical, but that assumption is likely to be wrong. This is because the bridge deck, on which the lower end of each hanger is attached, would probably prevent horizontal motion of the hangers from the prescribed co-ordinates x_j, $j = 1, 2, \ \ldots \ N_F$, more than the main cables would do.

It is obvious that it is necessary to calculate all s_j, $j = 1, 2, \ \ldots \ , N_F$, as well as the length of the unstrained cable L_0, in order to achieve the desired results for axially elastic cables. In the next chapter, we show how this can be done.

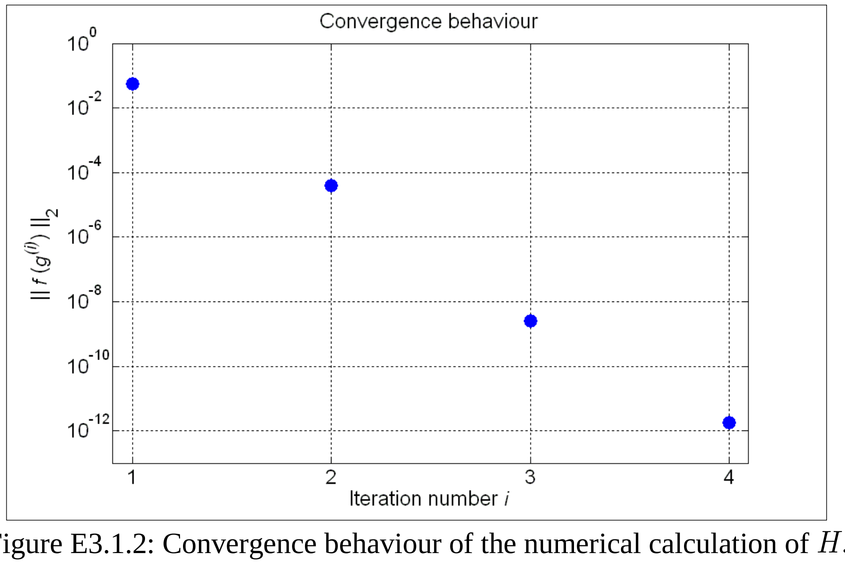


Figure E3.1.2: Convergence behaviour of the numerical calculation of H.

7. Shape control of cables loaded by gravity and vertical point forces

The equations for $T = f_T(s),\ x = f_x(s)$ and $z = f_z(s),$ derived in the previous chapter, require that L_0 and all $s_j,\ j = 1, 2,\ \ldots\ , N_F$, are known. However, in many applications, these parameters must be calculated in order to obtain the desired results. For instance, it may, for a specific load, be necessary to determine all $s_j,\ j = 1, 2,\ \ldots\ , N_F$, such that the x co-ordinates of the points of application of the point loads are equal to prescribed values and, at the same time, calculate L_0 such that the z co-ordinate of the lowest point of the cable centerline is equal to a prescribed value.

In this chapter, we describe how certain problems of this kind may be solved. It is then assumed that the shape of the cable is given by Equations (6.20) and (6.21) at known values of the external loads, of which gravity is always included.

First, the basic features of the method for solving this kind of problems are outlined, after which more detailed descriptions of some special problems are given.

Each problem is solved by creating an equation for each unknown parameter that is to be determined. Thus, a system of equations, with as many equations as there are unknown parameters, is obtained.

In every problem, the system of Equations is solved iteratively for the unknown parameters by using Newton's method. This involves calculation of the entries of the Jacobian-matrix, which are derivatives, with respect to the unknown parameters, of the functions f_x and f_z, see Equation (A.3). However, calculation of these derivatives may be an obstacle in some cases. This applies to problems where one is to calculate the co-ordinates $s = s_j,\ j = 1, 2,\ \ldots\ , N_F$, of the points of application of all point loads, in order that the x co-ordinates of these points will assume prescribed values. It is then necessary to compute the derivatives, with respect to the unknown variables, of f_x, at the initially unknown values of $s = s_i, j = 1, 2,\ \ldots\ , N_F$, which are at points where, at least, the derivatives with respect to s of f_x, are not defined. The key to the solution of such problems is found in the rule that decides how n is selected as a function of s in $x = f_x(s)$ and $z = f_z(s)$ of Equations (6.20) and (6.21) (see Figure 6.3 on page 53). According to this rule, the co-ordinates $x_j = f_x(s_j)$ and $z_j = f_z(s_j)$ constitute the first point of the continuous line segment that consist of the x and z co-ordinates given by $x = f_x(s)$ and $z = f_z(s)$ on the interval $s_j \leq s < s_{j+1}$. On this interval of s, both $x = f_x$ and $z = f_z$ are continuously differentiable with respect to the unknown variables and, consequently, Newton's method can be used in order to solve the equations.

In the present paper, the derivatives in the Jacobian-matrix are, for simplicity, calculated numerically by using forward differences according to Equation (A.4) or (A.5). This requires that the value of the increment h of Equation (A.4), or the increments $\Delta g_v,\ v = 1, 2,\ \ldots\ m,$ of Equation (A.5), are correctly chosen in order for the numerical solution to converge efficiently (see the examples below). Alternatively, the derivatives can be calculated analytically, but that is not done in the present paper.

7.1 Cables of symmetric shape

It is now assumed that the ends of the cable are on the same vertical level, and that the cable, with respect to its midpoint, is symmetrically loaded by N_F downwardly directed vertical point loads and gravity. Consequently, the shape of the cable is symmetric with respect to its midpoint.

We prescribe the x and z co-ordinates of points A and B. It is also required that the z co-ordinate of the lowest point of the cable centerline, as well as the x co-ordinate of every hanger centerline, are to be equal to prescribed values.

The shape of the cable is given by Equations (6.20) and (6.21), but it is not sufficient to determine H and V_A only. We must also determine L_0 and all s_j, $j = 1, 2, \ \ldots \ , N_F$, such that the requirements are fulfilled. Since the cable is of symmetric shape, it holds that the prescribed z co-ordinate of the lowest point of the cable centerline is given by Equation (6.21) at $s = L_0/2$. By creating an equation for each unknown parameter to be determined, we get a system of equations that, if N_F is an even number, can be written as

$$
\begin{aligned}
&0 = -x_{cB} + f_{x_c}(H, V_A, L_0, \boldsymbol{S_j})|_{s=L_0} \\
&0 = -z_{cB} + f_{z_c}(H, V_A, L_0, \boldsymbol{S_j})|_{s=L_0} \\
&0 = -z_{cD} + f_{z_c}(H, V_A, L_0, \boldsymbol{S_j^*})|_{s=L_0/2} \\
&0 = -x_{c1} + f_{x_c}(H, V_A, s_1)|_{s=s_1} \\
&0 = -x_{c2} + f_{x_c}(H, V_A, s_1, s_2)|_{s=s_2} \\
&0 = -x_{c3} + f_{x_c}(H, V_A, s_1, s_2, s_3)|_{s=s_3} \\
&\quad \vdots \\
&0 = -x_{cN_F} + f_{x_c}(H, V_A, \boldsymbol{S_j})|_{s=N_F},
\end{aligned}
\tag{7.1}
$$

where

$$
x_{cB} = x_B - x_A, \quad z_{cB} = z_B - z_A = 0, \quad z_{cD} = z_D - z_A,
$$

$$
x_{cj} = x_j - x_A, \ j = 1, 2, \ \ldots \ , N_F,
$$

$$
\boldsymbol{S_j^*} = \begin{pmatrix} s_1 & s_2 & \ldots & s_{(N_F/2)} \end{pmatrix}^T, \quad \boldsymbol{S_j} = \begin{pmatrix} s_1 & s_2 & \ldots & s_{N_F} \end{pmatrix}^T.
$$

In Equations (7.1), the functions f_{x_c} and f_{z_c} are given by Equations (6.20) and (6.21). The system of Equations (7.1) is solved for H, V_A, L_0 and all components of the matrix $\boldsymbol{S_j}$.

It is seen from the system of Equations (7.1) and other systems of equations given in this chapter, that it is the inserted value of the arc-length co-ordinate s that decides which s_j, $j = 1, 2, \ \ldots \ , N_F$, that the functions f_{x_c} and f_{z_c} are functions of, during the numerical calculation of the unknown parameters. The reason for this is that the functions f_{x_c} and f_{z_c} do not depend on any $s_j > s$, since the upper bound n of the summation operators Σ in Equations (6.20) and (6.21), is determined by the inserted value of s, as shown in Figure 6.3.

The fact that both geometry and loading are symmetric means that we can solve a smaller system of equations and, consequently, the computational time can be greatly reduced. If the symmetry properties are used, and if the number of point loads N_F is an even number, the system of equations to be solved, can be written as

$$
\begin{aligned}
0 &= -x_{cB} + f_{x_c}(H, V_A, L_0, \boldsymbol{S_j^*})|_{s=L_0} \\
0 &= -z_{cB} + f_{z_c}(H, V_A, L_0, \boldsymbol{S_j^*})|_{s=L_0} \\
0 &= -z_{cD} + f_{z_c}(H, V_A, L_0, \boldsymbol{S_j^*})|_{s=L_0/2} \\
0 &= -x_{c1} + f_{x_c}(H, V_A, s_1)|_{s=s_1} \\
0 &= -x_{c2} + f_{x_c}(H, V_A, s_1, s_2)|_{s=s_2} \\
0 &= -x_{c3} + f_{x_c}(H, V_A, s_1, s_2, s_3)|_{s=s_3} \\
&\vdots \\
0 &= -x_{cN_F/2} + f_{x_c}(H, V_A, \boldsymbol{S_j^*})|_{s=s_{(N_F/2)}},
\end{aligned}
\tag{7.2}
$$

where

$$x_{cB} = x_B - x_A, \quad z_{cB} = z_B - z_A = 0, \quad z_{cD} = z_D - z_A,$$

$$x_{cj} = x_j - x_A, \; j = 1, 2, \; \dots \; , \frac{N_F}{2},$$

$$\boldsymbol{S_j^*} = \begin{pmatrix} s_1 & s_2 & \dots & s_{(N_F/2)} \end{pmatrix}^T.$$

Equations (7.2) are solved numerically for H, V_A, L_0 and all s_j, $j = 1, 2, \; \dots \; , N_F/2$. All the remaining s_j, $(N_F/2 + 1) \le j \le N_F$, are, in every iteration during the numerical solution process, calculated according to

$$s_{N_F+1-j} = L_0 - s_j, \; 1 \le j \le \frac{N_F}{2}, \; 0 < s_j < \frac{L_0}{2}, \tag{7.3}$$

before they are inserted into the first and second of Equations (7.2).

7.2 Cables of asymmetric shape

Shape control can also be performed for cables of asymmetric shape. This may concern sections I and III of the main cables of a suspension bridge (see Figure 3.3b on page 19). For such a bridge, we may for section III of the main cables, require that the x and z co-ordinates of point F, defined in Figure 7.1, are to assume prescribed values, in order to make sure that the main cables are located above the bridge deck. If the co-ordinates of points A and F of this section of the main cables are prescribed, only the x or z co-ordinate of point B can be prescribed.

Analogous requirements can be placed on the co-ordinates of certain points on the cable centerline of section I of the main cables of a bridge of Type 2.

We now consider a problem in which H, the x and z co-ordinates of points A and F, the x co-ordinate of point B, and the x co-ordinates of all hanger centerlines are prescribed. In this problem, the shape of the cable is determined by calculating the unknown parameters V_A, L_0, s_F and $\boldsymbol{S_j} = \begin{pmatrix} s_1 & s_2 & \dots & s_{N_F} \end{pmatrix}^T$. The system of equations to be solved is given by

$$
\begin{aligned}
0 &= -x_{cF} + f_{x_c}(V_A, s_F, \boldsymbol{S_j})|_{s=s_F} \\
0 &= -z_{cF} + f_{z_c}(V_A, s_F, \boldsymbol{S_j})|_{s=s_F} \\
0 &= -x_{cB} + f_{z_c}(V_A, L_0, \boldsymbol{S_j})|_{s=L_0} \\
0 &= -x_{c1} + f_{x_c}(V_A, s_1)|_{s=s_1} \\
0 &= -x_{c2} + f_{x_c}(V_A, s_1, s_2)|_{s=s_2} \\
0 &= -x_{c3} + f_{x_c}(V_A, s_1, s_2, s_3)|_{s=s_3} \\
&\vdots \\
0 &= -x_{cN_F} + f_{x_c}(V_A, L_0, \boldsymbol{S_j})|_{s=N_F},
\end{aligned}
\tag{7.4}
$$

where

$$
x_{cF} = x_F - x_A, \quad z_{cF} = z_F - z_A, \quad x_{cB} = x_B - x_A,
$$

$$
x_{cj} = x_j - x_A, \; j = 1, 2, \; \dots \;, N_F, \quad \boldsymbol{S_j} = \begin{pmatrix} s_1 & s_2 & \dots & s_{N_F} \end{pmatrix}^T.
$$

As another example of cables of asymmetric shape, we take cables for which it holds that the minimum point of their centerline is also the point with horizontal tangent vector. This means that point D is located between points A and B (see Figure 7.2), and that the co-ordinates s_D and $x_D = f_x(s_D)$ of point D are unknown. Using the theory described in Chapter 6, it may not be possible to require that the lowest point of the cable centerline is also to be the point where the tangent vector is horizontal, as is done by the forth of Equations (5.6) in Chapter 5. This is due to the fact that, according to the cable theory, there may not be any point where the tangent vector is horizontal, since the derivatives of $x = f_x(s)$ and $z = f_z(s)$, with respect to s, are discontinuous functions of s on the interval $0 \leq s \leq L_0$.

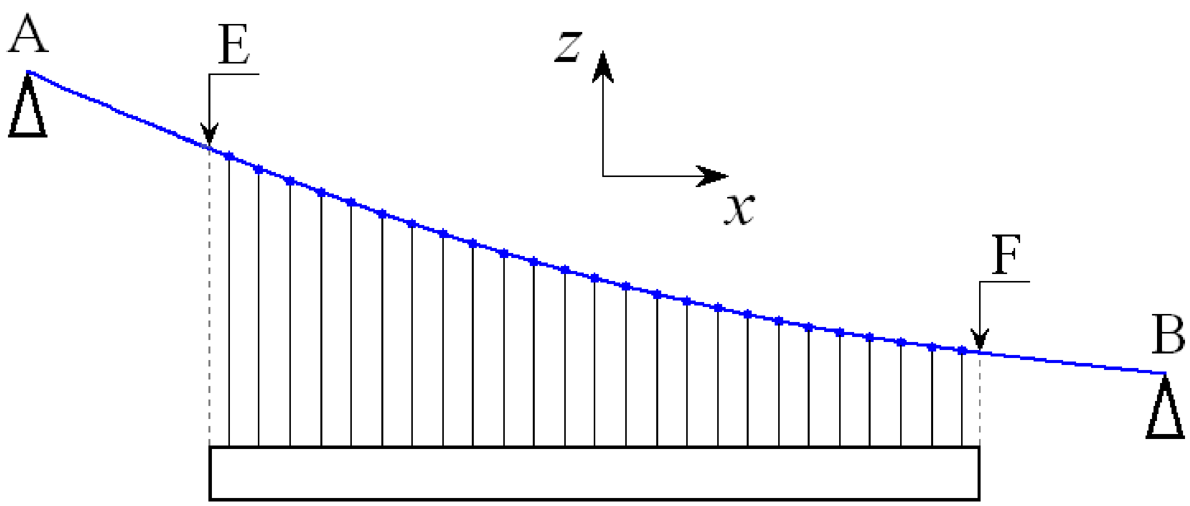


Figure 7.1: Points E and F are, respectively, defined as the points on the cable centerline that have the same x co-ordinate as the respective ends of the load that is carried by the cable. The co-ordinates of point E are denoted $(x_E, z_E) = (f_x(s_E), f_z(s_E))$, whereas those of point F are denoted $(x_F, z_F) = (f_x(s_F), f_z(s_F))$.

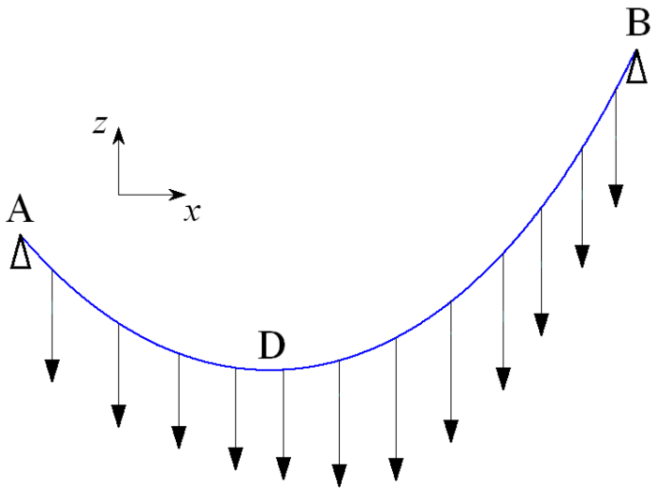


Figure 7.2: A cable of asymmetric shape, loaded by gravity and vertical point forces.

If point D is located sufficiently far away from the nearest hanger, then, in some cases, the shape of the cable can be successfully controlled in a straightforward way. If this is the case, and we have prescribed the x and z co-ordinates of points A and B, the x co-ordinates of all hanger centerlines, and the z co-ordinate of the lowest point of the cable centerline z_D, then the shape of the cable can be determined by calculating H, V_A, L_0, s_D and all components of the matrix $\boldsymbol{S_j} = \begin{pmatrix} s_1 & s_2 & \dots & s_{N_F} \end{pmatrix}^T$. This is done by solving the following system of equations for the unknown parameters:

$$
\begin{aligned}
0 &= -x_{cB} + f_{x_c}(H, V_A, L_0, \boldsymbol{S_j})|_{s=L_0} \\
0 &= -z_{cB} + f_{z_c}(H, V_A, L_0, \boldsymbol{S_j})|_{s=L_0} \\
0 &= -z_{cD} + f_{z_c}(H, V_A, s_D, \boldsymbol{S_j^{\#}}|_{s=s_D} \\
0 &= \frac{df_z}{ds}(H, V_A, s_D)|_{s=s_D} \\
0 &= -x_{c1} + f_{x_c}(H, V_A, s_1)|_{s=s_1} \\
0 &= -x_{c2} + f_{x_c}(H, V_A, s_1, s_2)|_{s=s_2} \\
0 &= -x_{c3} + f_{x_c}(H, V_A, s_1, s_2, s_3)|_{s=s_3} \\
&\ \vdots \\
0 &= -x_{cN_F} + f_{x_c}(H, V_A, \boldsymbol{S}_j)|_{s=s_{N_F}},
\end{aligned}
\tag{7.5}
$$

where

$$x_{cB} = x_B - x_A, \quad z_{cB} = z_B - z_A, \quad z_{cD} = z_D - z_A,$$

$$x_{cj} = x_j - x_A, \ j = 1, 2, \ \dots \ , N_F,$$

$$\boldsymbol{S_j^{\#}} = \begin{pmatrix} s_1 & s_2 & \dots & s_{\#} \end{pmatrix}^T, \quad \boldsymbol{S_j} = \begin{pmatrix} s_1 & s_2 & \dots & s_{N_F} \end{pmatrix}^T,$$

and $s_j = s_{\#}$ is the largest of all $s_j < s_D, \ j = 1, 2, \ \dots \ , N_F$. The derivative in Equation $(7.5)_4$ is calculated according to Equation (6.28).

If, on the other hand, there, according to the cable theory, is no point on the cable centerline with horizontal tangent vector, then Equation $(7.5)_4$ is invalid, and, consequently, the system of equations (7.5) cannot be solved. Whether Equations (7.5) can be solved or not depends to a high degree on the prescribed x co-ordinates of the hanger centerlines. In problems where the hangers are required to be equally spaced horizontally, the prescribed x co-ordinate of each hanger centerline is often calculated as

$$x_j^{\text{pres}} = x_A + \frac{\Delta x}{2} + \Delta x(j-1), \quad j = 1, 2, \ \ldots \ , N_F,$$

where

$$\Delta x = \frac{x_B - x_A}{N_F}.$$

The solvability of Equations (7.5) is, therefore, highly dependent on the number of hangers N_F.

If it is not possible to choose N_F such that Equations (7.5) can be solved, one might try to overcome this difficulty by replacing the condition $\left.\frac{df_z}{ds}\right|_{s=s_D} = 0$, imposed by Equation $(7.5)_4$, with the condition that the derivative, with respect to s, of the z co-ordinate of a spline curve, is to be equal to zero at a certain value of s. This spline curve is located in the sz-plane, and the z co-ordinates of the spline curve are assumed to be given by the function $z = f_z^{\text{spl}}(s)$. In every iteration of the numerical solution procedure, a new spline curve is created, and each spline curve is based on the knots given by

$$(0, z_A), \quad (s_j, f_z(s_j)), \quad (L_0, z_B), \quad j = 1, 2, \ \ldots \ , N_F, \tag{7.6}$$

available at iteration i. The function $z = f_z^{\text{spl}}(s)$ is differentiated analytically, with respect to s, in order to create the equation

$$0 = \left.\frac{df_z^{\text{spl}}}{ds}\right|_{s=s_{\text{mp}}^{\text{apx}}}, \tag{7.7}$$

where $s = s_{\text{mp}}^{\text{apx}}$ is the approximate value of the parameter $s = s_{\text{mp}}$, at which the minimum point of the function $z = f_z(s)$ is obtained. We use the parameter $s_{\text{mp}}^{\text{apx}}$ in Equation (7.7) instead of s_{mp}, because the obtained solution will be approximate as far as the parameter s_{mp} is concerned. Since the minimum point of the function $z = f_z(s)$ is not necessarily a point with horizontal tangent vector, the minimum point of this function is called z_{mp} instead of z_D.

The derivative df_z^{spl}/ds in Equation (7.7) is a function of several parameters during the numerical solution process; see Equation $(7.8)_4$. As described before, it is the inserted value of the arc-length co-ordinate s that decides which of the parameters s_j, $j = 1, 2, \ \ldots \ , N_F$, that the functions f_x and f_z are functions of, during the calculation of the unknown parameters. The fact that Equation (7.7) contains the derivative of the z co-ordinate of a spline curve, infers that the dependence, or lack of dependence, on the parameters s_j, $j = 1, 2, \ \ldots \ , N_F$, of the function df_z^{spl}/ds, is governed by a complicated s-dependent relation. However, by calculating the derivatives of df_z^{spl}/ds numerically in the solution process, the s-dependent relation does not have to be investigated and, for simplicity, the parameters that the function

df_z^{spl}/ds currently depends on, are in the system of Equations (7.8) below, thought to be collected in the matrix $\boldsymbol{S_j^{\mathrm{spl}}}$.

With prescribed x and z co-ordinates of points A and B, prescribed x coordinates of all hanger centerlines, and prescribed value of the z coordinate of the lowest point of the cable centerline $z_{\mathrm{mp}}^{\mathrm{req}}$, we can determine the shape of the cable by calculating the values of $H,\ V_A,\ L_0, s_{\mathrm{mp}}^{\mathrm{apx}}$ and all components of the matrix $\boldsymbol{S_j} = \begin{pmatrix} s_1 & s_2 & \dots & s_{N_F}\end{pmatrix}^T$. The system of equations to be solved in this case can be written as

$$\begin{aligned}
0 &= -x_{cB} + f_{x_c}(H, V_A, L_0, \boldsymbol{S_j})|_{s=L_0}\\
0 &= -z_{cB} + f_{z_c}(H, V_A, L_0, \boldsymbol{S_j})|_{s=L_0}\\
0 &= -z_{c,\mathrm{mp}}^{\mathrm{req}} + f_{z_c}(H, V_A, s_{\mathrm{mp}}^{\mathrm{apx}}, \boldsymbol{S_j^{\#}})|_{s=s_{\mathrm{mp}}^{\mathrm{apx}}}\\
0 &= \frac{df_z^{\mathrm{spl}}}{ds}(H, V_A, s_{\mathrm{mp}}^{\mathrm{apx}}, \boldsymbol{S_j^{\mathrm{spl}}})|_{s=s_{\mathrm{mp}}^{\mathrm{apx}}}\\
0 &= -x_{c1} + f_{x_c}(H, V_A, s_1)|_{s=s_1}\\
0 &= -x_{c2} + f_{x_c}(H, V_A, s_1, s_2)|_{s=s_2}\\
0 &= -x_{c3} + f_{x_c}(H, V_A, s_1, s_2, s_3)|_{s=s_3}\\
&\ \ \vdots\\
0 &= -x_{cN_F} + f_{x_c}(H, V_A, \boldsymbol{S}_j)|_{s=s_{N_F}},
\end{aligned} \tag{7.8}$$

where

$$x_{cB} = x_B - x_A, \quad z_{cB} = z_B - z_A, \quad z_{c,\mathrm{mp}}^{\mathrm{req}} = z_{\mathrm{mp}}^{\mathrm{req}} - z_A,$$

$$x_{cj} = x_j - x_A,\ j = 1, 2,\ \dots\ , N_F,$$

$$\boldsymbol{S_j^{\#}} = \begin{pmatrix} s_1 & s_2 & \dots & s_{\#}\end{pmatrix}^T, \quad \boldsymbol{S_j} = \begin{pmatrix} s_1 & s_2 & \dots & s_{N_F}\end{pmatrix}^T,$$

and $s_j = s_{\#}$ is the largest of all $s_j < s_{\mathrm{mp}}^{\mathrm{apx}},\ j = 1, 2,\ \dots\ , N_F$.

An important feature of the obtained solution to Equations (7.8) is that the solution accurately fulfills the prescribed values of all $x_j,\ j = 1, 2,\ \dots\ N_F$. Had this not been so, the solution would be useless in applications where it is necessary to be able to predict the magnitude of each point load applied to the cable (see the end of Example 3.1 for a description of such a situation).

A consequence of the fact that the fourth of Equations (7.8) is a derivative of the z co-ordinate of a spline curve, instead of the function $z = f_z(s)$, is that the calculated value $s_{\mathrm{mp}}^{\mathrm{apx}}$ may not be equal to s_{mp}, which is the value of s that minimizes the function $z = f_z(s)$. However, if the difference between the prescribed value of $z_{\mathrm{mp}} = f_z(s_{\mathrm{mp}})$ and the value given by $z = f_z(s_{\mathrm{mp}}^{\mathrm{apx}})$, is sufficiently small, the obtained solution is acceptable.

In the present paper, we determine the value s_{mp}, of the obtained solution, by using the MATLAB-function fminbnd, which finds the minimum of a function without computing any derivatives.

It is possible that other curve fitting techniques are better suited for this kind of problem than spline interpolation, but that is not investigated in this paper.

7.3 Concluding remarks

As stated before, the cable theory of Irwin and Sinclair [2] model the derivatives $\dfrac{dx}{ds} = \dfrac{df_x}{ds}(s)$, $\dfrac{dz}{ds} = \dfrac{df_z}{ds}(s)$, and $\dfrac{dz}{dx} = \dfrac{dz}{ds} \cdot \dfrac{ds}{dx}$, as discontinuous at the points of application of the point loads. This means that the there is a jump in the graphs of the above given derivative functions at every s_j, $j = 1, 2, \ \ldots \ , N_F$. If the jumps of the derivatives are large, the theory of Irwin and Sinclair [2] is probably only realistic in applications involving thin cables of negligible flexural rigidity. In the examples of the present chapter, we solve shape control problems that concern cables, which are used in applications where the used cable theory predicts that the jumps of the derivatives are quite small, see, for example, Figures E3.1.6 to E3.1.8 in Example 3.1, III. We therefore assume that the results given in the examples are realistic. The method presented in the present chapter works well numerically, also in problems where the jumps of the derivatives are considerably larger than those shown in the present chapter. This can, for instance, be seen by setting $N_F = 2$ in Example 3.1, III.

Once one of the systems of equations (7.1), (7.2), (7.4), (7.5) or (7.8) are solved, we compare the results provided by the functions $x = f_x(s)$ and $z = f_z(s)$, with the prescribed values that these functions are required to provide at certain points. This is done by calculating the difference between the prescribed values of the Cartesian co-ordinates and the values obtained from the functions $x = f_x(s)$ and $z = f_z(s)$, according to Equations (5.8) and (5.9):

$$x^{\text{diff}} = x^{\text{pres}} - f_x(s) \qquad \text{(5.8) repeated}$$

$$z^{\text{diff}} = z^{\text{pres}} - f_z(s) \qquad \text{(5.9) repeated}$$

Example 3.1, III (continued from Chapter 6)

Calculate the parameters $H,\ V_A,\ L_0$ and $\boldsymbol{S_j} = \begin{pmatrix} s_1 & s_2 & \dots & s_{N_F} \end{pmatrix}^T$, such that the x and z co-ordinates of the lowest point of the cable centerline are equal to

$$x_D^{\mathrm{pres}} = f_x(L_0/2) = 650 \text{ m}, \tag{E3.1.6}$$

$$z_D^{\mathrm{pres}} = f_z(L_0/2) = 0 \text{ m}, \tag{E3.1.7}$$

and such that the x co-ordinate of each respective hanger centerline is equal to

$$x_j^{\mathrm{pres}} = f_x(s_j) = x_A + \frac{\Delta x}{2} + \Delta x(j-1), \quad j = 1, 2, \ \dots\ , N_F. \tag{E3.1.8}$$

Also compute the length L of the strained cable, the elongation ΔL of the cable, and the cable tension T.

For convenience, we repeat some relevant parameter values:

$$A_0 = 0.6 \text{ m}^2, \quad E = 2.05 \cdot 10^{11} \ \text{ N/m}^2, \quad \rho = 7850 \text{ kg/m}^3, \quad q_L = 1.1 \cdot 10^5 \text{ N/m},$$

$$(x_A, z_A) = (0, 150) \text{ m}, \quad (x_B, z_B) = (1300, 150) \text{ m}, \quad \Delta x = 16.25 \text{ m}, \quad N_F = 80,$$

$$F_j = 1.788 \cdot 10^6 \text{ N}, \ j = 1, 2, \ \dots\ , N_F, \quad H_{\mathrm{ie}} = 2.200 \cdot 10^8 \text{ N}, \quad L_{\mathrm{ie}} = 1344.781 \text{ m}.$$

Solution

Since the shape and loading of the cable are symmetric, we solve Equations (7.1) for the unknown parameters. For comparison of results, we also solve Equations (7.2).

The starting values for H, L_0 and V_A are given by

$$H_0 = H_{\mathrm{ie}}, \quad L_{0,0} = L_{\mathrm{ie}},$$

$$V_{A,0} = \frac{1}{2}\left(W_{\mathrm{ie}} + \sum_{j=0}^{80} F_j\right) = 1.026 \cdot 10^8 \text{ N},$$

whereas Equation (3.16) is used to calculate the starting values $s_{j,0}$ at x_j^{pres}, $j = 1, 2, \ \dots\ ,$ N_F. The systems of Equations (7.1) and (7.2) are then solved to give

$$H = 2.20998 \cdot 10^8 \text{ N}, \quad V_A = 1.02512 \cdot 10^8 \text{ N}, \quad L_0 = 1342.374 \text{ m}.$$

All $s_j,\ j = 1, 2, \ \dots\ , N_F$, are given in Appendix C.

Figure E3.1.3 shows the shape of the cable, and Figures E3.1.5 and E3.1.6 show the convergence behaviour of the numerical solution processes.

By using Equations (5.8) and (5.9), we investigate how well Equations (6.20) and (6.21) satisfiy requirements (E3.1.6) to (E3.1.8). The result of these calculations, concerning Equations (7.1), are given by

$$x_D^{\text{diff}} = x_D^{\text{pres}} - f_x(L_0/2) = 4.5 \cdot 10^{-13} \text{ m},$$

$$z_D^{\text{diff}} = z_D^{\text{pres}} - f_z(L_0/2) = 0 \text{ m},$$

$$\max(||x_j^{\text{diff}}||_2) = \max(||x_j^{\text{pres}} - f_x(s_j)||_2) = 2.7 \cdot 10^{-12} \text{ m}, \ j = 1, 2, \ \ldots \ , N_F,$$

whereas for Equations (7.2), the results are as follows:

$$x_D^{\text{diff}} = x_D^{\text{pres}} - f_x(L_0/2) = -1.3 \cdot 10^{-12} \text{ m},$$

$$z_D^{\text{diff}} = z_D^{\text{pres}} - f_z(L_0/2) = -9.1 \cdot 10^{-12} \text{ m},$$

$$\max(||x_j^{\text{diff}}||_2) = \max(||x_j^{\text{pres}} - f_x(s_j)||_2) = 5.5 \cdot 10^{-12} \text{ m}, \ j = 1, 2, \ \ldots \ , N_F.$$

It is clearly seen that the shape control was successful in both cases.

As far as we know, the main benefit of solving Equations (7.2) instead of Equations (7.1) is, at least in this problem, that the solution process is significantly less time consuming. It is likely that the solution can be obtained in less than one second using a reasonably fast home computer. However, if a fast computer is used, there might not be any reason for solving Equations (7.2), since programming of Equations (7.1) is more straightforward.

For the numerical solution of Equations (7.1) and (7.2), the derivatives in the Jacobian-matrix were calculated using $h = 0.1$ in Equation (A.4). However, the results obtained by solving Equations (7.2) can be slightly improved by using the increments $\Delta H = \Delta V_A = 0.1$ and $\Delta L_0 = \Delta s_j = 10^{-8}$, $j = 1, 2, \ \ldots \ , N_F$, in Equation (A.5).

Equation (6.31) gives the length of the strained cable:

$$L = 1344.871 \text{ m}.$$

The cable has, as a result of the loading by gravity and the external forces, lengthened

$$\Delta L = L - L_0 = 2.497 \text{ m}.$$

The cable tension $T = f_T(s)$ versus $x = f_x(s)$ is shown in Figure E3.1.4, whereas various derivatives of the co-ordinates x and z are depicted in Figures E3.1.7 to E3.1.9.

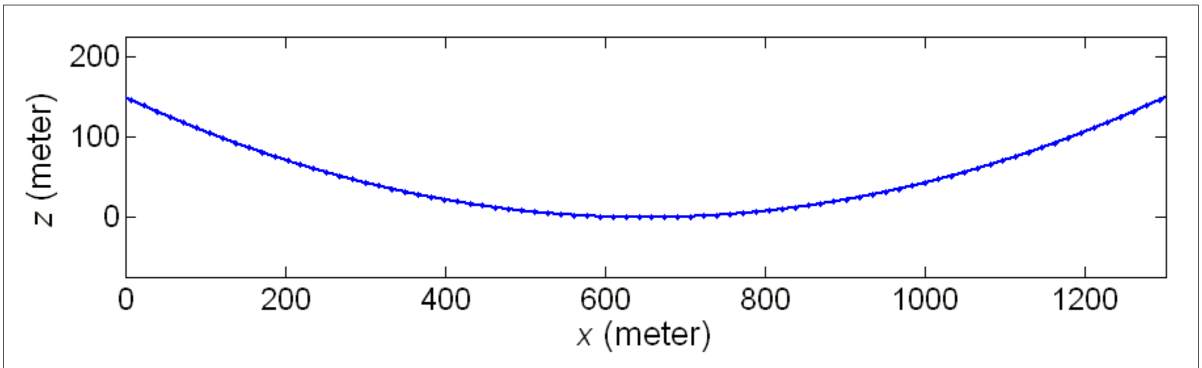


Figure E3.1.3: The shape of the cable. The dots indicate the locations of the points of application of the vertical external point loads.

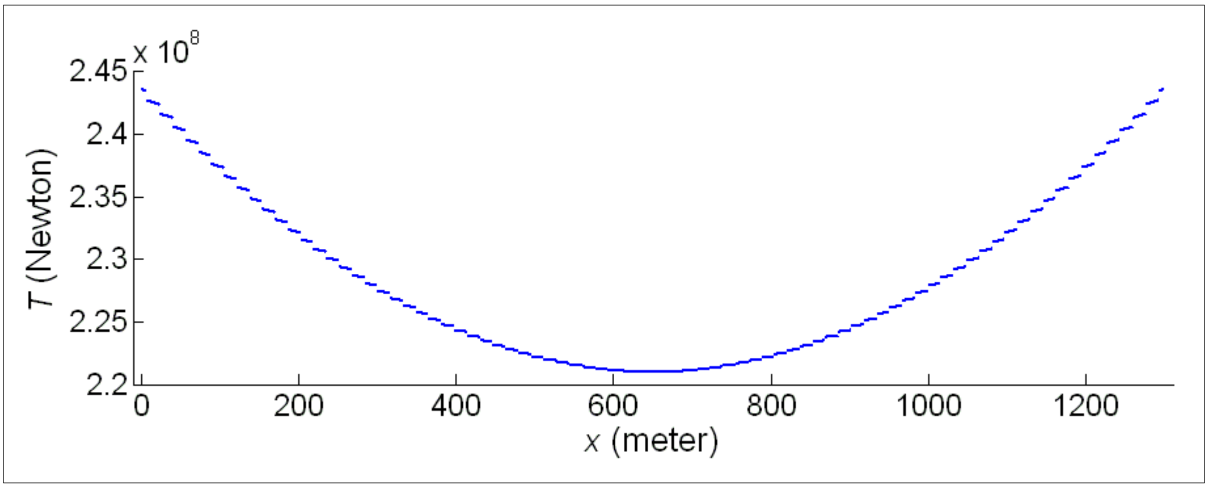


Figure E3.1.4: The co-ordinate $x = f_x(s)$ versus the cable tension $T = f_T(s)$.

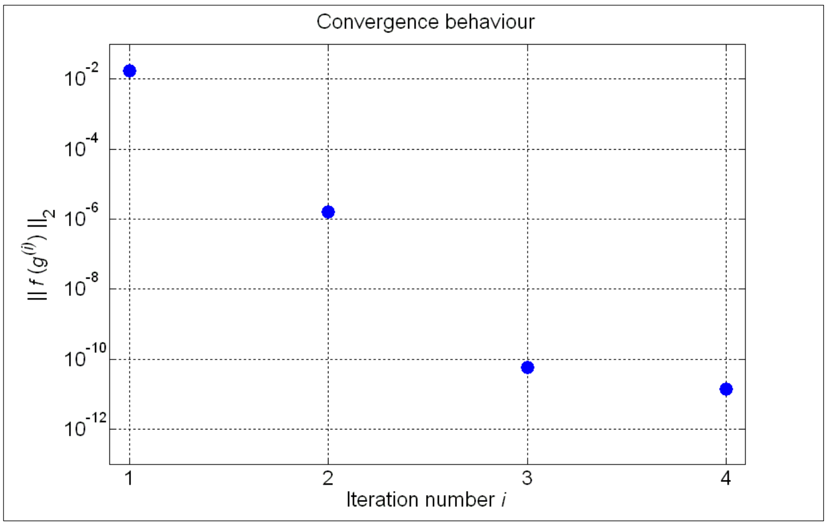


Figure E3.1.5: Convergence behaviour of the numerical solution of Equations (7.1).

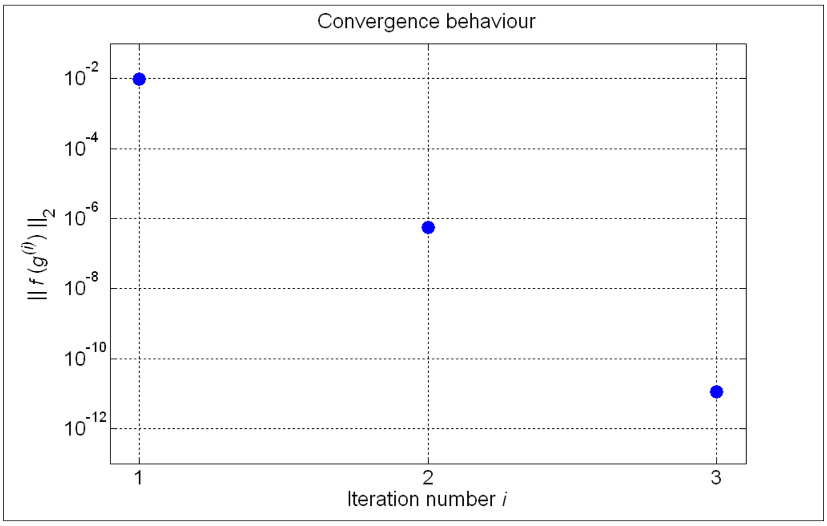


Figure E3.1.6: Convergence behaviour of the numerical solution of Equations (7.2). The convergence of the numerical solution of Equations (7.2) is, in this problem, faster than the numerical solution of Equations (7.1).

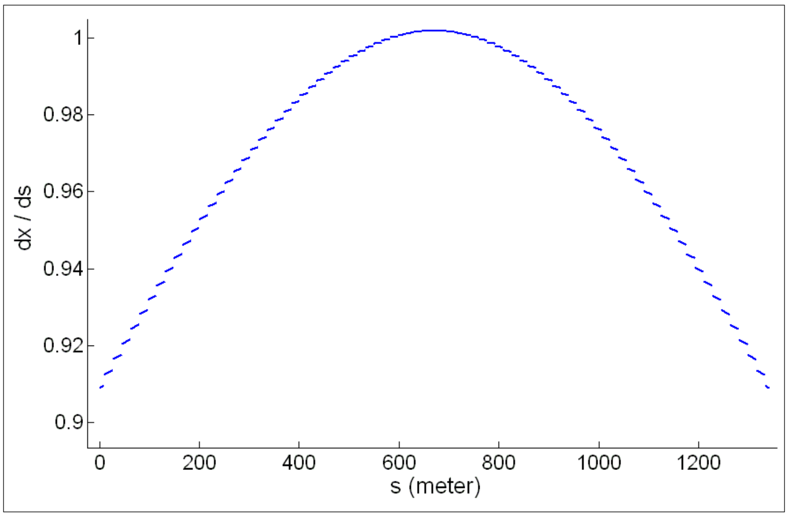


Figure E3.1.7: The co-ordinate s versus the derivative $\frac{dx}{ds}$.

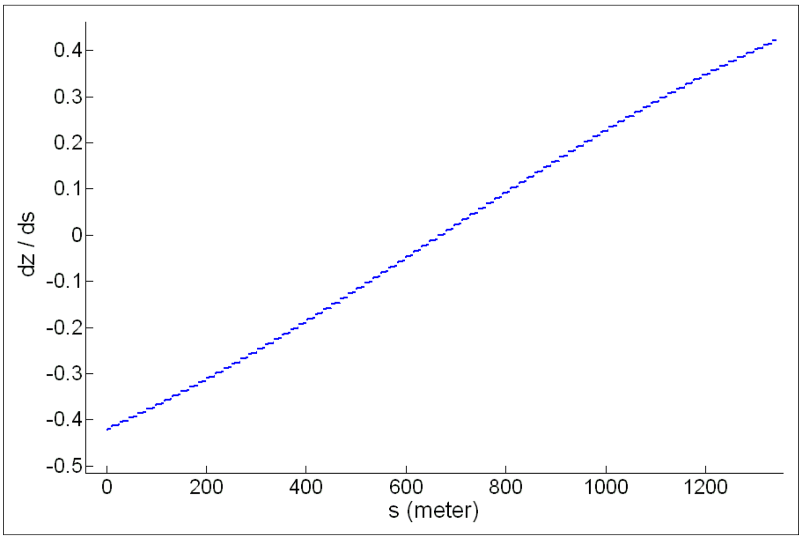


Figure E3.1.8: The co-ordinate s versus the derivative $\frac{dz}{ds}$.

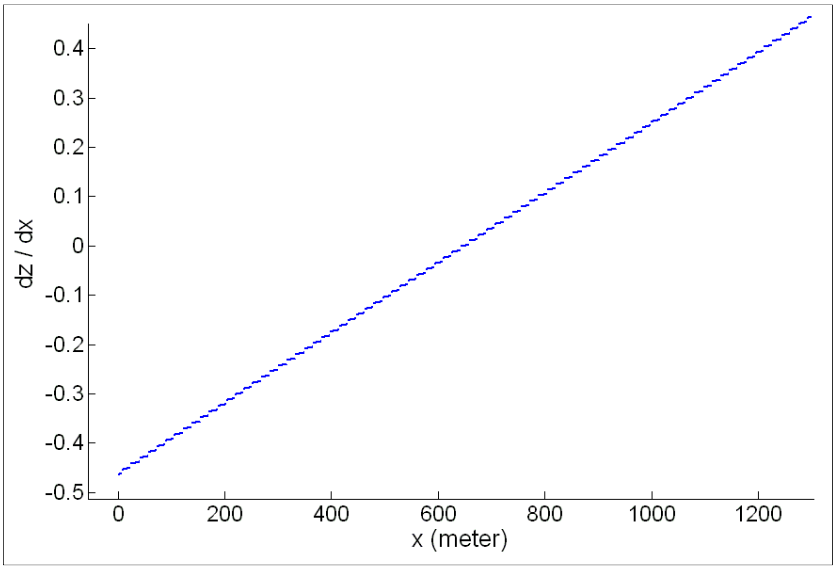


Figure E3.1.9: The co-ordinate $x = f_x(s)$ versus the derivative $\frac{dz}{dx} = \frac{dz}{ds}\frac{ds}{dx}$. As the jumps of the derivatives are small, we assume that the obtained solution is realistic.

Example 3.2, II (continued from Chapter 3)

In this example, we analyse Section III of each main cable of the symmetric suspension bridge that is described in Example 3.1. We then assume that the bridge is of Type 2 (see Figure 3.3). It is required that the horizontal component H of the cable tension T, is to be equal to

$$H = 2.20998 \cdot 10^8 \text{ N},$$

as calculated in Example 3.1, III.

The properties of the main cables, the external load q_L, and the co-ordinate system, are the same as in Example 3.1, III.

The prescribed x and z co-ordinates of the end points of the analysed cable section are equal to

$$(x_A, z_A) = (1310, 150) \text{ m} \quad \text{and} \quad x_B = 1800 \text{ m}.$$

We assume that section III of each main cable carry half of the portion of the bridge deck that stretches horizontally from $x = x_E = x_A$ to $x = x_F^{\text{pres}}$ (see Equation (E3.2.1)) via $N_F = 25$, equally spaced hangers.

The external loads on the main cables, other than gravity, are applied as vertical point forces representing the forces from the hangers.

Since the hangers are assumed to be equally spaced, we calculate the horizontal distance between the centerlines of two adjacent hangers as

$$\Delta x = \frac{x_F^{\text{pres}} - x_E}{N_F} = \frac{x_F^{\text{pres}} - x_A}{N_F} = \frac{1700 - 1310}{25} = 15.6 \text{ m}.$$

For the given values of H, x_A, z_A and x_B, calculate V_A, L_0, s_F and all s_j, $j = 1, 2, \ldots, N_F$, such that the x and z co-ordinates of point F are equal to

$$x_F^{\text{pres}} = f_x(s_F) = 1700 \text{ m}, \tag{E3.2.1}$$

$$z_F^{\text{pres}} = f_z(s_F) = 2 \text{ m}, \tag{E3.2.2}$$

and such that the x co-ordinate of each respective hanger centerline is equal to

$$x_j^{\text{pres}} = f_x(s_j) = x_A + \frac{\Delta x}{2} + \Delta x(j-1), \quad j = 1, 2, \ldots, N_F, \tag{E3.2.3}$$

Also calculate the length of the strained cable and its elongation.

Solution

It is assumed that each hanger carries an equally large portion of the bridge deck, and that the weight of such a portion is $q_L \Delta x$. Consequently, the magnitude of the tensile forces $\boldsymbol{F}_n$, $n = 1, 2, \ldots, N_F$, sustained by each hanger, is calculated as

$$F_j = q_L \Delta x = 1.716 \cdot 10^6 \text{ N}, \quad j = 1, 2, \ \ldots \ , N_F,$$

where the self-weight per unit length of each main cable is given by

$$q_c = A_0 \rho g = 4.621 \cdot 10^4 \text{ N/m}.$$

We obtain the solution to this problem by solving the system of Equations (7.4) for V_A, L_0, s_F and $\boldsymbol{S_n}$, which gives

$$V_A = 1.151 \cdot 10^8 \text{ N}, \quad L_0 = 519.781 \text{ m}, \quad s_F = 417.385 \text{ m}.$$

All $s_j,\ j = 1, 2,\ \ldots\ , N_F$, are given in Appendix C. These results are obtained by using $h = 0.1$ in Equation (A.4).

As starting values for the numerical solution process, we used

$$s_{F,0} = L_{\text{ie}} = 418.146 \text{ m},$$

$$L_{0,0} = L_{\text{ie}} + (x_B - x_F^{\text{pres}}) = 418.146 + (1800 - 1700) = 518.146 \text{ m},$$

$$V_{A,0} = V_{A,\text{ie}} = \sqrt{(T_{A,\text{ie}})^2 - H^2} = \sqrt{(2.488 \cdot 10^8)^2 - H^2} = 1.143 \cdot 10^8 \text{ N},$$

whereas Equation (3.16) is used to calculate the starting values $s_{j,0}$ at $x_j^{\text{pres}},\ j = 1, 2,\ \ldots\ , N_F$.

The shape of the cable is shown in Figure E3.2.3, and the convergence behaviour of the numerical solution process is shown in Figure E3.2.6.

By using Equations (5.8) and (5.9), we investigate how accurately Equations (6.20) and (6.21) agrees with requirements (E3.2.1) to (E3.2.3). This gives

$$x_F^{\text{diff}} = x_F^{\text{pres}} - f_x(s_F) = 4.5 \cdot 10^{-13} \text{ m},$$

$$z_F^{\text{diff}} = z_F^{\text{pres}} - f_z(s_F) = 0 \text{ m},$$

$$\max(||x_j^{\text{diff}}||_2) = \max(||x_j^{\text{pres}} - f_x(s_j)||_2) = 1.6 \cdot 10^{-12} \text{ m},\ j = 1, 2,\ \ldots\ , N_F.$$

We calculate the length of the strained cable according to Equation (6.31):

$$L = 520.774 \text{ m}.$$

Consequently, the elongation of the cable is equal to

$$\Delta L = L - L_0 = 0.993 \text{ m}.$$

Figure E3.2.4 shows the cable tension which is calculated by using Equation (6.19), whereas Figure E3.2.5 depicts the co-ordinate $x = f_x(s)$ versus the derivative $\dfrac{dz}{dx} = \dfrac{dz}{ds}\dfrac{ds}{dx}$, computed according to Equations (6.27) and (6.28).

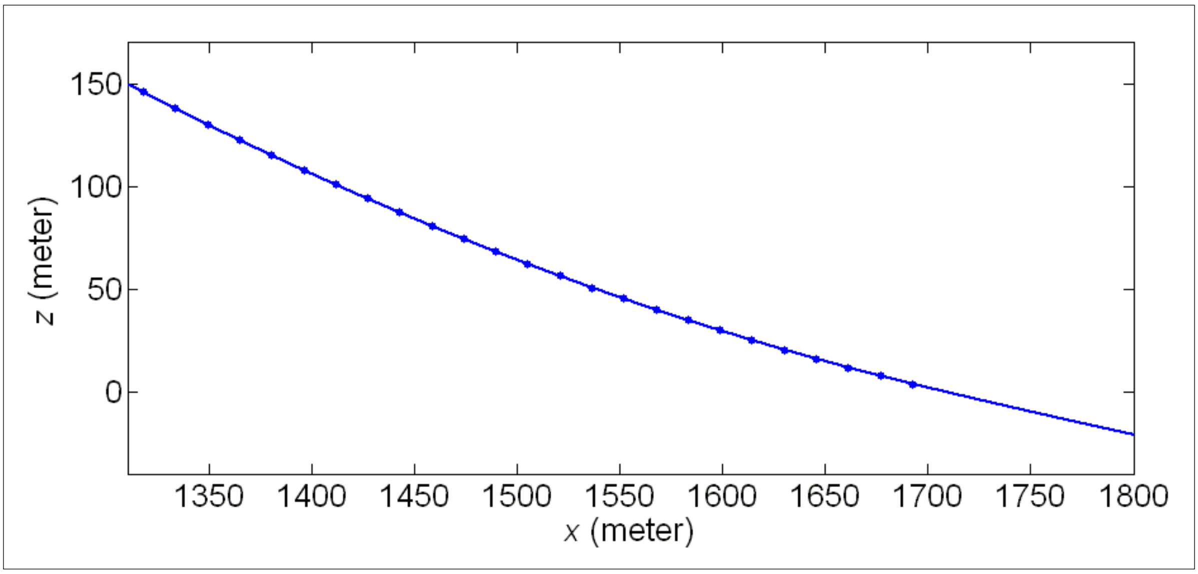


Figure E3.2.3: The shape of the cable. The dots indicate the locations of the points of application of the vertical external point loads.

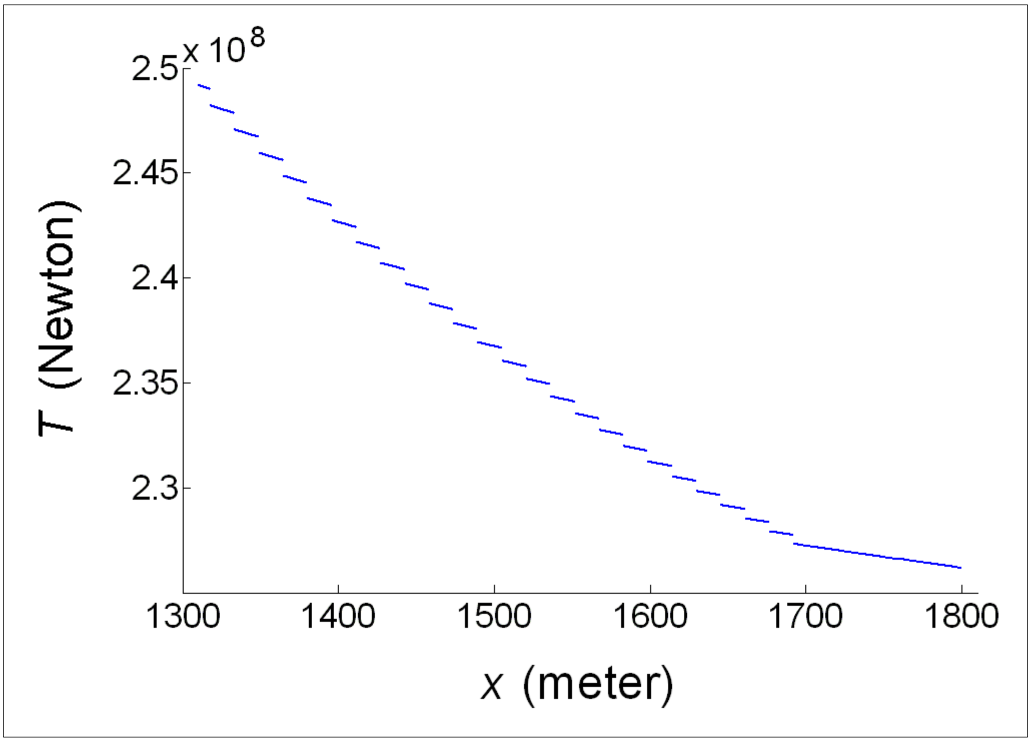


Figure E3.2.4: The co-ordinate $x = f_x(s)$ versus the cable tension $T = f_T(s)$.

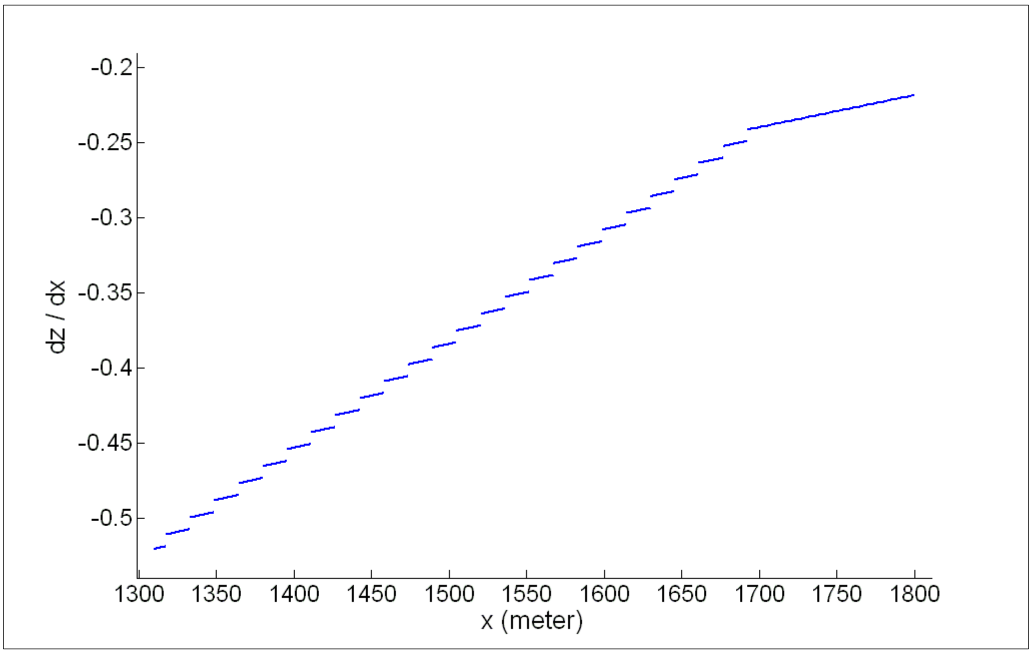


Figure E3.2.5: The co-ordinate $x = f_x(s)$ versus the derivative $\frac{dz}{dx} = \frac{dz}{ds}\frac{ds}{dx}$. As seen in the figure, the jumps of the derivatives are small, and, therefore, we assume that the obtained solution is realistic.

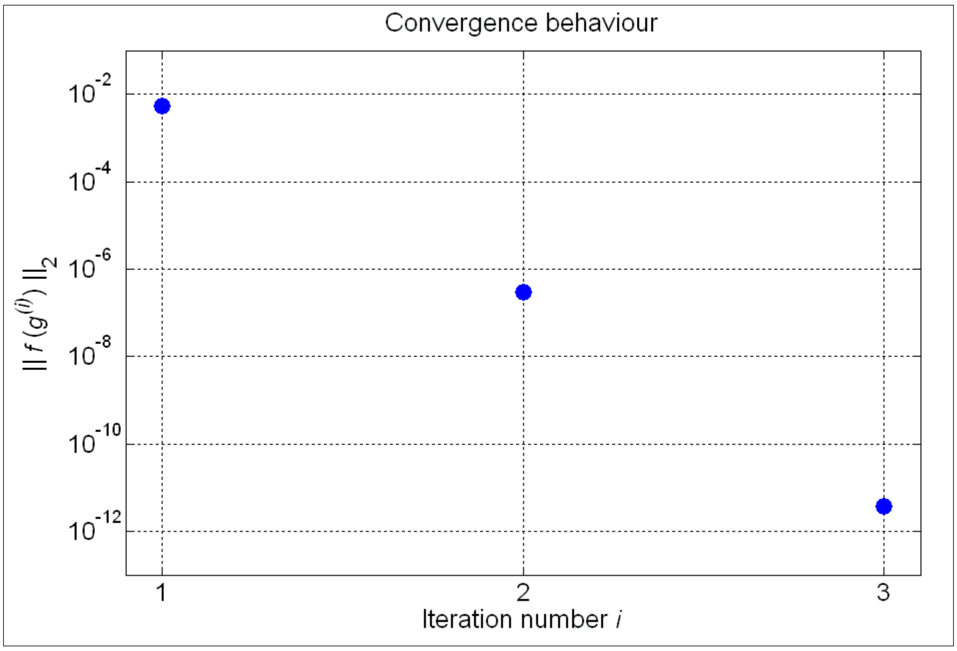


Figure E3.2.6: Convergence behaviour of the numerical calculation of V_A, L_0, s_F and $\boldsymbol{S_n}$. These parameters are calculated with $h = 0.1$ in Equation (A.4).

Example 3.3, II (continued from Chapter 3)

We now include axial elasticity in the analysis according to the theory described in Chapter 6. The external loads on the main cables, other than gravity, are applied as vertical point forces representing the forces from the hangers.

As stated in Part I of this example, the main span section of each main cable has a span of $d_h = 1164$ m, and is expected to have a sag $d_v = 200$ m, when no vehicles are on the bridge.

When unloaded, the cross sectional area of each main cable is $A_0 = 0.7\ \mathrm{m}^2$. We assume that the cable material is linearly elastic with modulus of elasticity $E = 2.05 \cdot 10^{11}$ N/m^2, and density $\rho = 7850$ kg/m^3. Consequently, the axial rigidity of each main cable is $EA_0 = 1.435 \cdot 10^{11}$ N.

The road deck is horizontal, and its constant weight per unit length along the span is $q_{L,\text{total}} = 3 \cdot 10^5$ N/m. Each main cable carries half the weight of the road deck, which means that

$$q_L = \frac{q_{L,\text{total}}}{2} = 1.5 \cdot 10^5\ \mathrm{N/m}.$$

The weight of the main span section of the bridge deck is assumed to be transferred to each main cable via $N_F = 75$ vertical and equally spaced hangers of negligible weight, along the main span of the bridge.

Since the hangers are assumed to be equally spaced, the horizontal distance between the centerlines of two adjacent hangers is calculated as

$$\Delta x = \frac{x_B - x_A}{N_F} = \frac{1164}{75} = 15.52\ \mathrm{m}.$$

It is also assumed that the towers are rigid and that the main cables cannot move relative to the towers.

The subsequent analysis concerns the main span section of each main cable, which is assumed to carry half of the portion of the bridge deck that stretches horizontally from $x = x_A$ to $x = x_B$.

Calculate the parameters $H,\ V_A,\ L_0,\ s_D$ and $\boldsymbol{S_j} = \begin{pmatrix} s_1 & s_2 & \dots & s_{N_F} \end{pmatrix}^T$, such that the z co-ordinate of the lowest point of the cable centerline is equal to

$$z_D^{\text{pres}} = f_z(s_D) = 0\ \mathrm{m}, \tag{E3.3.1}$$

and such that the x co-ordinate of each respective hanger centerline is equal to

$$x_j^{\text{pres}} = f_x(s_j) = x_A + \frac{\Delta x}{2} + \Delta x(j-1), \quad j = 1, 2,\ \dots\ , N_F. \tag{E3.3.2}$$

Solve the problem for a few different values of N_F.

For $N_F = 75$, compute also the length L of the strained cable, the elongation ΔL of the cable and the cable tension T.

For convenience, the values of some parameters that are relevant for the inextensible cable are repeated here:

$$H_{\text{ie}} = 2.370 \cdot 10^8\ \mathrm{N}, \quad L_{\text{ie}} = 1214.794\ \mathrm{m}, \quad x_{D,\text{ie}} = 482.145\ \mathrm{m},$$

$$T_{A,\text{ie}} = 2.566 \cdot 10^8 \text{ N}, \quad T_{B,\text{ie}} = 2.748 \cdot 10^8 \text{ N}, \quad s_{D,\text{ie}} = f_{s,\text{ie}}(x_{D,\text{ie}}) = 495.635 \text{ m},$$

$$V_{A,\text{ie}} = ((T_{A,\text{ie}})^2 - H_{\text{ie}}^2)^{1/2} = 9.831 \cdot 10^7 \text{ N}.$$

Solution

It is assumed that each hanger carries an equally large portion of the bridge deck, and that the weight of such a portion is $q_L \Delta x$. Consequently, the magnitude of the tensile forces $\boldsymbol{F}_j$, $j = 1, 2, \ \ldots \ , N_F$, sustained by each hanger, is calculated as

$$F_j = q_L \Delta x = 2.328 \cdot 10^6 \text{ N}, \quad j = 1, 2, \ \ldots \ , N_F,$$

where the self-weight per unit length of each main cable is equal to

$$q_c = A_0 \rho g = 5.391 \cdot 10^4 \text{ N/m}.$$

Since the co-ordinates of points A and B are given by

$$(x_A, z_A) = (0, 100) \text{ m}, \quad \text{and} \quad (x_B, z_B) = (1164, 200) \text{ m},$$

and since the lowest point of the cable centerline is prescribed to be located below both points A and B, it holds that the shape of the cable is asymmetric, and that the lowest point of the cable centerline is a point with horizontal tangent vector. For a given value of N_F, it is, therefore, not known beforehand whether the theory of Irvine and Sinclair [2] is able to model the lowest point of the cable centerline as a point with horizontal tangent vector. However, it turns out that for $N_F = 75$, the theory is able to do so, and, consequently, we solve Equations (7.5) for the unknown parameters, which gives

$$H = 2.383 \cdot 10^8 \text{ N}, \qquad V_A = 9.910 \cdot 10^7 \text{ N},$$

$$L_0 = 1212.820 \text{ m}, \qquad s_D = 499.556 \text{ m}.$$

All $s_j, \ j = 1, 2, \ \ldots \ , N_F$, are given in Appendix C. The numerical solution procedure was initiated with the starting values $V_{A,0} = V_{A,\text{ie}}$, $H_0 = H_{\text{ie}}$, $L_{0,0} = L_{\text{ie}}$, $s_{D,0} = s_{D,\text{ie}}$, whereas we calculated every $s_{j,0}$, $j = 1, 2, \ \ldots \ , N_F$, according to Equation (3.16) at x_j^{pres}. Figure E3.3.3 shows the shape of the cable, and the convergence behaviour of the solution procedure is shown in Figure E3.3.8a.

An investigation of the quality, with respect to requirements (E3.3.1) and (E3.3.2), of the obtained solution to this problem, reveals that

$$z_D^{\text{diff}} = z_D^{\text{pres}} - f_z(s_D) = 3.6 \cdot 10^{-12} \text{ m},$$

$$\max(||x_j^{\text{diff}}||_2) = \max(||x_j^{\text{pres}} - f_x(s_j)||_2) = 1.6 \cdot 10^{-12} \text{ m}, \ j = 1, 2, \ \ldots \ , N_F.$$

Equation (6.31) gives the length of the strained cable:

$$L = 1214.925 \text{ m}.$$

With this information, the elongation of the cable can be calculated, which gives

$$\Delta L = L - L_0 = 2.105 \text{ m}.$$

Figure E3.3.7 shows the cable tension.

Equations (7.5) can only be solved if the cable theory is able to model the minimum point of the cable centerline as a point with horizontal tangent vector. For $N_F = 75$, this is possible since the derivative $dz/ds = 0$ at $s = s_D$, as shown in Figure E3.3.5.

Figures E3.3.4a and E3.3.4b, respectively, show the derivatives dx/ds and dz/ds, whereas Figure E3.3.6 shows the derivative dz/dx. Since the jumps of the derivatives are small, we expect that the obtained solution is realistic.

If the number of vertical point loads is changed to $N_F = 74$ or $N_F = 76$, Equations (7.5) cannot be solved, at least not if the numerical solution procedure is initiated with the same starting values as were used for $N_F = 75$. For example, it can be mentioned that for all values of N_F in the interval $70 \leq N_F \leq 90$, Equations (7.5) can be solved successfully for $N_F = 70,\ 75,\ 77,\ 82,\ 87,\ 89$, wheras Equations (7.8) can be solved successfully for all N_F in the above mentioned interval.

The number of point loads is now changed to $N_F = 73$. Therefore, Equations (7.8) are solved instead of Equations (7.5), which gives

$$H = 2.383 \cdot 10^8 \text{ N}, \qquad V_A = 9.910 \cdot 10^7 \text{ N},$$

$$L_0 = 1212.818 \text{ m}, \qquad s_{\text{mp}}^{\text{apx}} = 495.299 \text{ m}.$$

Figure E3.3.8b shows the convergence behaviour of the numerical solution process.

By using the MATLAB-function fminbnd, we get

$$s_{\text{mp}} = 498.975 \text{ m},$$

which is the value of s that minimizes the function $z = f_z(s)$.

Since, in this case, the lowest point of the cable centerline may not be the point with horizontal tangent vector, requirement (E3.3.1) is replaced by the requirement

$$z_{\text{mp}}^{\text{pres}} = f_z(s_{\text{mp}}) = 0 \text{ m}. \tag{E3.3.3}$$

Thereby, the obtained solution can be analysed with respect to how well it complies with requirements (E3.3.3) and (E3.3.2), which gives

$$z_{\text{mp}}^{\text{diff}} = z_{\text{mp}}^{\text{pres}} - f_z(s_{\text{mp}}) = 0.0084 \text{ m},$$

$$\max(||x_j^{\text{diff}}||_2) = \max(||x_j^{\text{pres}} - f_x(s_j)||_2) = 2.3 \cdot 10^{-12} \text{ m},\ j = 1, 2,\ \ldots\ , N_F.$$

The value of $z_{\text{mp}}^{\text{diff}}$ given above for $N_F = 73$, is considerably larger than the value of z_D^{diff} for $N_F = 75$, but this value of $z_{\text{mp}}^{\text{diff}}$ might nevertheless be acceptable in an application since the solution of Equations (7.5) complies well with the requirements placed on the x co-ordinates of the hanger centerlines.

For all N_F in the interval $70 \leq N_F \leq 90$, it holds that

$$2.5 \cdot 10^{-7} \le ||z_{\mathrm{mp}}^{\mathrm{diff}}||_2 \le 0.0084 \text{ m}.$$

There does not appear to be any problems with the numerical solution procedure that concern solving Equations (7.8), at least not in the tests that were performed.

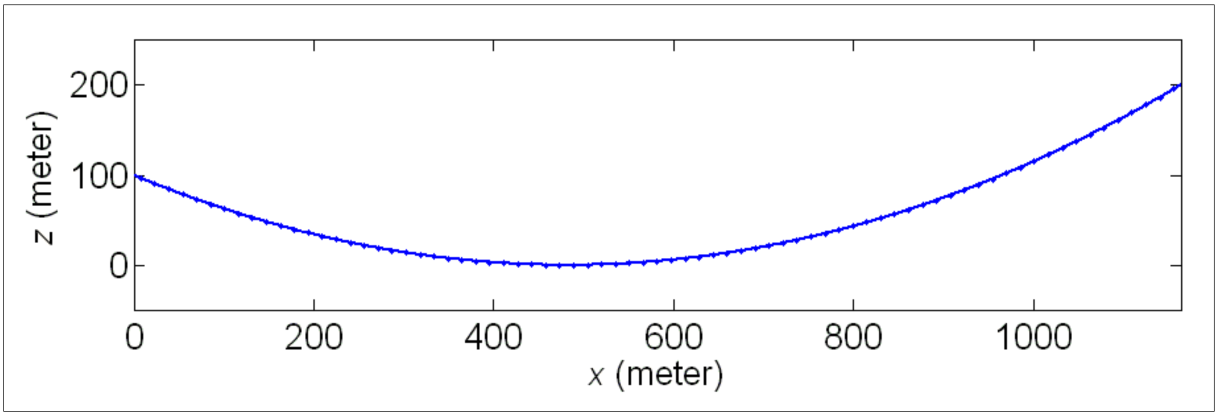


Figure E3.3.3: The shape of the cable. The dots indicate the locations of the points of application of the vertical external point loads.

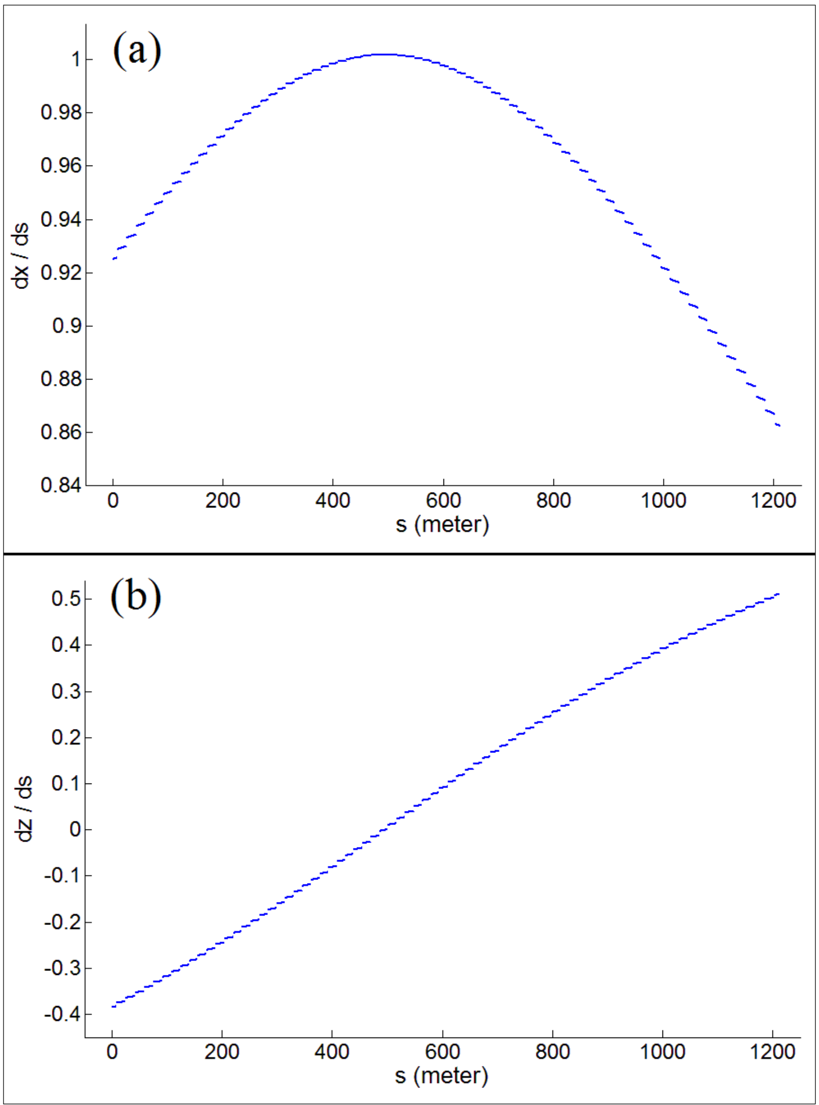


Figure E3.3.4: The co-ordinate s versus $\frac{dx}{ds}$ (a), and $\frac{dz}{ds}$ (b).

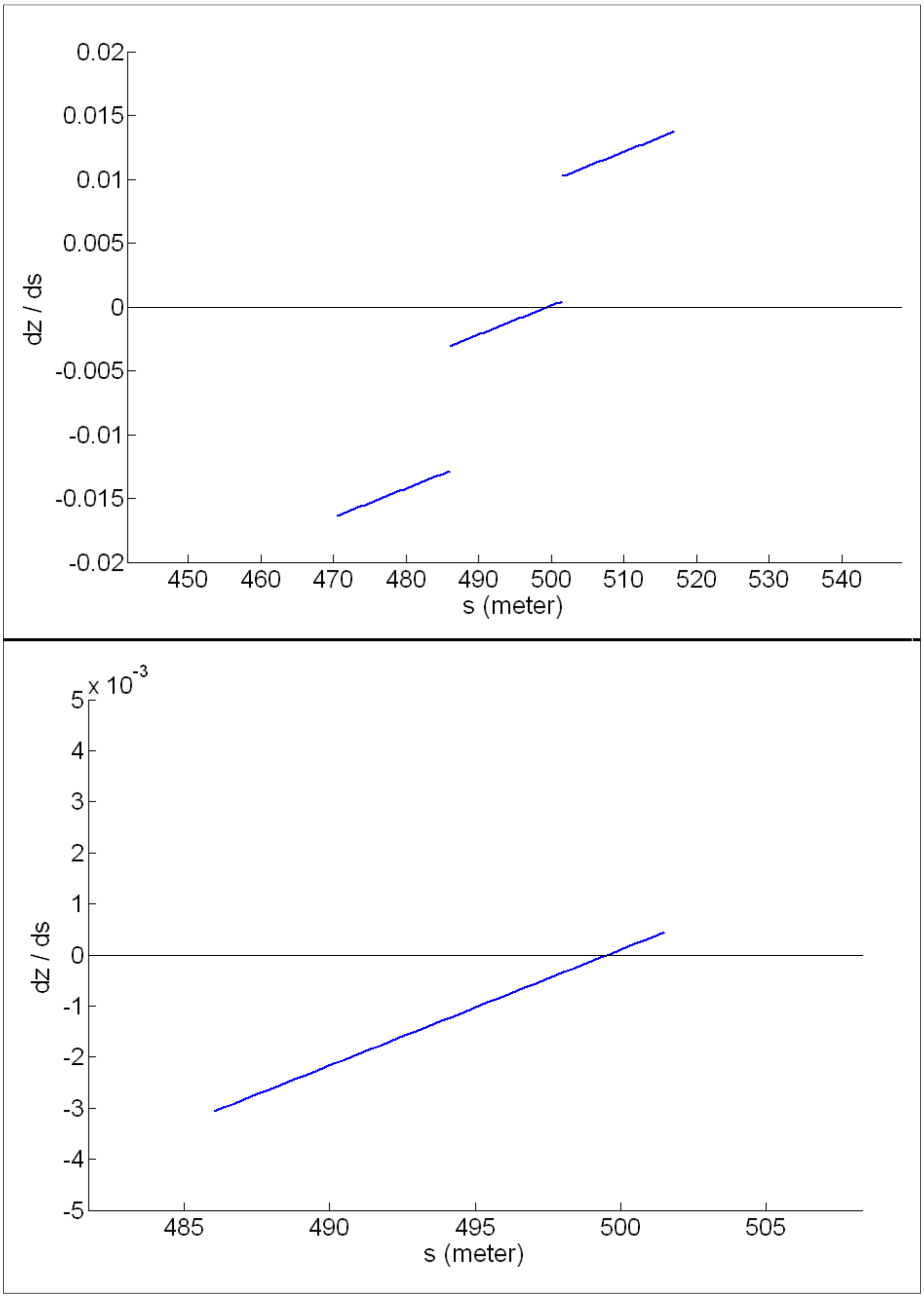


Figure E3.3.5: Close-ups of the co-ordinate s versus the derivative $\frac{dz}{ds}$. For the case that $N_F = 75$, the theory of Irvine and Sinclair [2] predicts that the lowest point of the cable centerline, is a point with horizontal tangent vector since $\frac{dz}{ds} = 0$ at $s = s_D$. Consequently, Equations (7.5) can be solved.

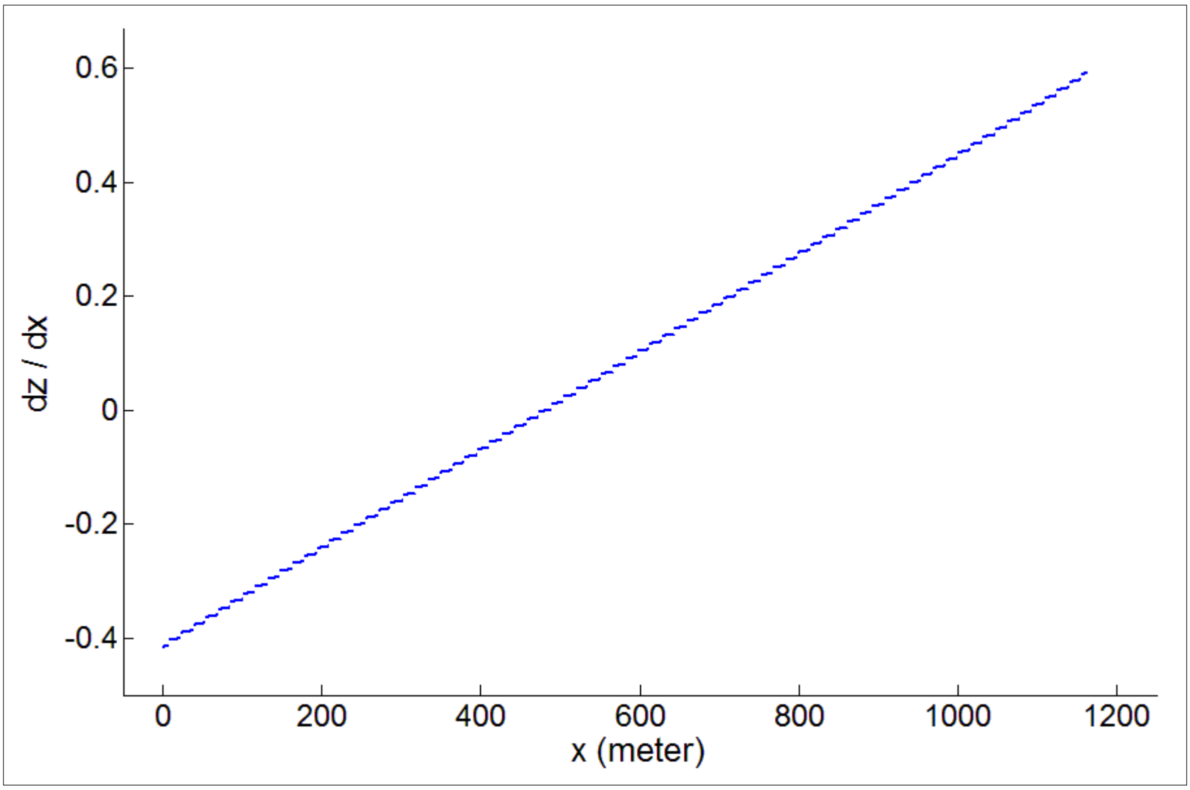
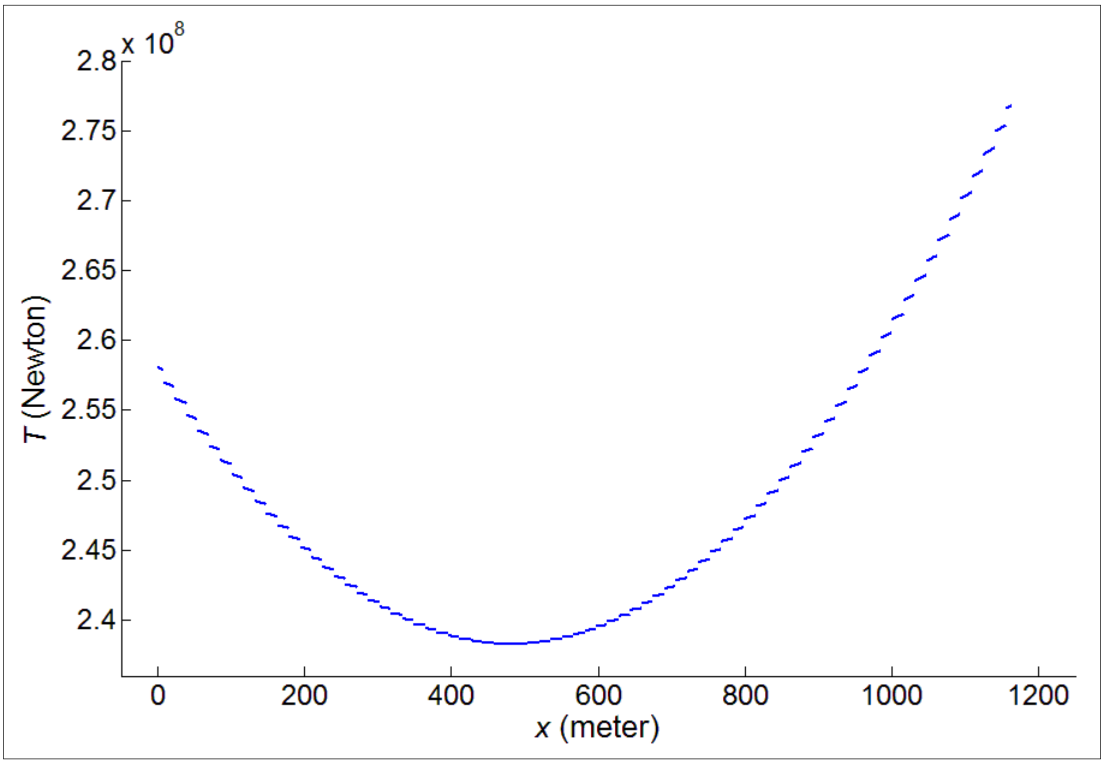


Figure E3.3.6: The co-ordinate $x = f_x(s)$ versus the derivative $\frac{dz}{dx} = \frac{dz}{ds}\frac{ds}{dx}$. Due to the fact that the jumps of the derivatives are small, we assume that the obtained solution is realistic.



E3.3.7: The cable tension T is plotted against the co-ordinate $x = f_x(s)$.

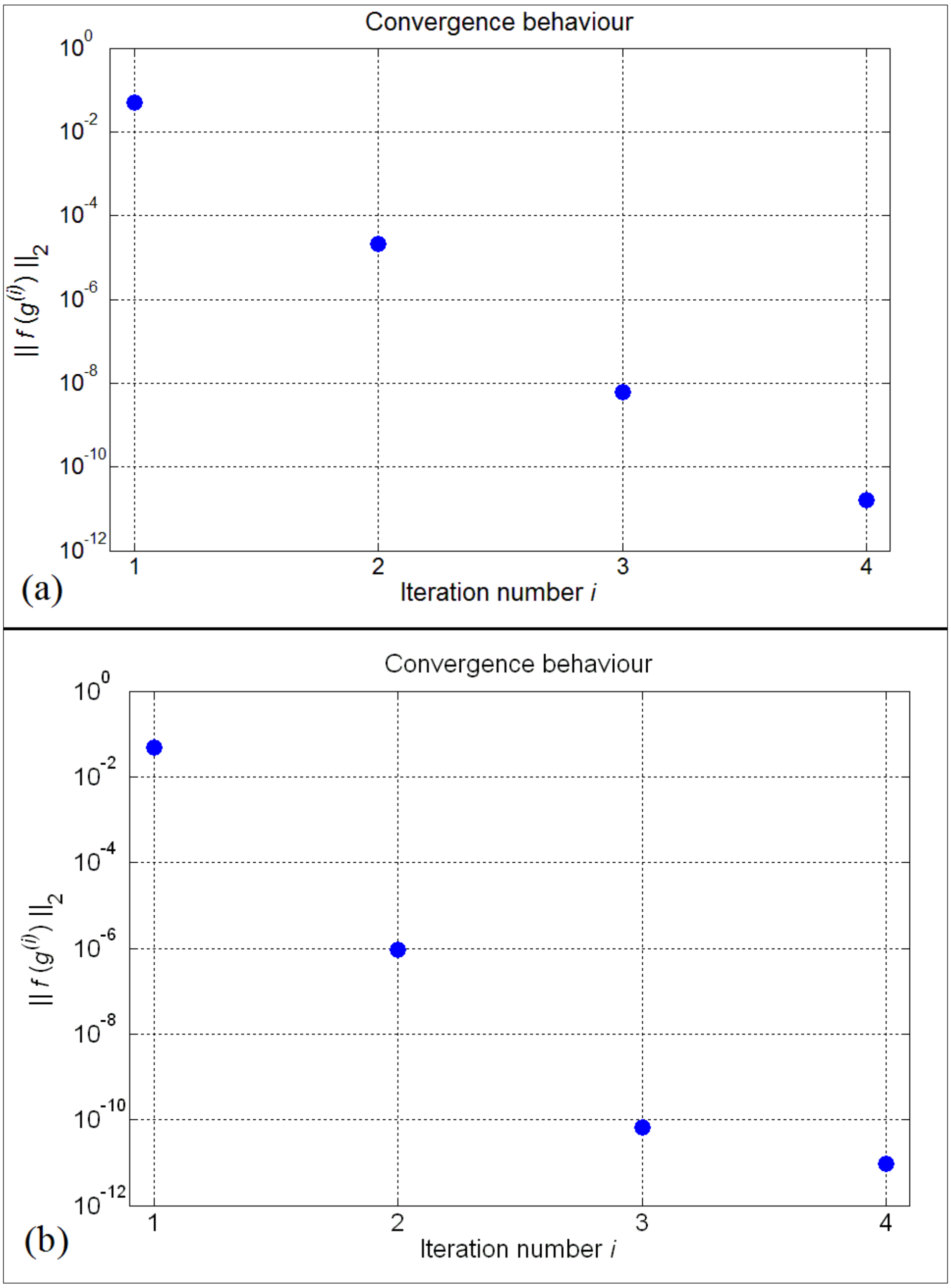


Figure E3.3.8: Convergence behaviour of the numerical solution procedure concerning (a) Equations (7.5), and (b) Equations (7.8). In case (a), the derivatives in the Jacobian-matrix were calculated using the increments $\Delta H = \Delta V_A = 0.1$ and $\Delta L_0 = \Delta s_D = \Delta s_j = 10^{-8}$, $j = 1, 2, \ \ldots \ , N_F$, in Equation (A.5). It is also possible to use the increment $h = 0.1$ in Equation (A.4), but the convergence is slightly faster if the abobe given increments are used in Equation (A.5). In case (b), $h = 0.1$ were used in Equation (A.4).

8. Remarks on flexibly supported cables

So far, we have assumed that every cable under consideration is rigidly supported at its ends. In reality, the supports will deflect when they are acted on by the forces applied by the cables. Consequently, geometric parameters such as the span d_h, the sag d_v and the co-ordinate z_D, may deviate significantly from those that would have been obtained with rigid supports. In some cases, the deflections of the supports are so small that they can be neglected. In other cases, it is necessary to consider the effects of support flexibility, since even relatively small deflections of the supports may considerably alter the value of certain parameters.

In some problems, the solutions obtained under the assumption of rigid supports may be useful even if the support deflections are significant. This applies to problems where it is possible to adjust the dimensions and location of the supports, in order to ensure that the supports deform in such a way that the end points, of the centerline of each cable section, are located at prescribed positions. It is then assumed that each section of the cables is under the same loads that were assumed to be applied in a previously solved shape control problem. For example, we consider a suspension bridge with main cables whose shape has been controlled under the assumption that the end points of sections I, II and III of the main cables are rigidly supported. As shown in Figure 8.1, the flexible supports are acted on by forces of the same magnitude, but of opposite direction, as the reaction forces on the ends of the respective cable section. For simplicity, only the right hand tower and sections II and III of the main cables are shown. The external loads on each main cable section are, for simplicity, illustrated by a fictitious load, which for section II is written as

$$F_{\text{load,II}} = q_c L_{0,II} + \sum_{j=0}^{N_{F,II}} F_j, \tag{8.1}$$

and analoguously for sections I and III.

Figure 8.1 shows that the forces $\boldsymbol{V}_{B,II}$ and $\boldsymbol{V}_{A,III}$, of magnitude $V_{B,II}$ and $V_{A,III}$, respectively, act to compress the right hand tower the distance

$$\Delta z_{\text{tower}} = z_{\text{top}} - z_{\text{top,0}}, \tag{8.2}$$

where $z_{\text{top,0}}$ is the z co-ordinate that the top of the right hand tower would be located at if $V_{B,II} = V_{A,III} = 0$.

Our objective is to calculate the value of $z_{\text{top,0}}$ such that z_{top} is equal to a prescribed value. As an example, the vertical deflection of each tower of a suspension bridge is expected to be governed by a linear spring model according to

$$\boldsymbol{F} = F\boldsymbol{e}_z = k(z_{\text{top}} - z_{\text{top,0}})\boldsymbol{e}_z, \tag{8.3}$$

where $\boldsymbol{F}$ is the force that act on the spring, F is the scalar component of $\boldsymbol{F}$, $k > 0$ is the spring constant, and $\boldsymbol{e}_z$ is a unit vector in the positive z-direction.

It is assumed that the shape of sections I, II and III of the main cables have been controlled such that H is the same in all sections of the main cables. The shape control problems were solved under the condition that the x and z co-ordinates of the ends of each section of the main cables are prescribed. The value of z_{top} is prescribed such that the ends of the centerline of every section of the main cables, will be located at the prescribed z co-

ordinates when the main cables are flexibly supported. For example, z_{top} of the tower shown in Figure 8.1 may be prescribed according to

$$z_{\text{top}} = z_{B,II} + a = z_{A,III} + a, \tag{8.4}$$

where $z_{B,II}$ and $z_{A,III}$ are the prescribed z co-ordinates of points B_{II} and A_{III}, respectively, and a is a constant which is used to compensate for the thickness of the main cable. Since the ends of all sections of the flexibly supported main cables are to be located in the same positions as prescribed for the shape control problem, the shape of every section of the main cables, and the reaction forces at the ends of each cable section, will be the same as those calculated in the shape control problem. We therefore calculate $z_{\text{top},0}$ from Equation (8.3) as

$$z_{\text{top},0} = z_{\text{top}} - \frac{F}{k} = z_{\text{top}} + \frac{V_{B,II} + V_{A,III}}{k} \tag{8.5}$$

(see Figure 8.1).

It may also be necessary to consider the deformations of the anchor blocks. Assuming that the anchor block, shown in Figure 8.1, is loaded by the forces $\boldsymbol{H}$ and $\boldsymbol{V}_{B,III}$ (probably not applied as point loads on the anchor block model), the corresponding deformations of the anchor block can be calculated. Once the deformations of the anchor blocks are known, it may then be possible to build each anchor block in position such that, for the loads assumed in the shape control problem, point A of section I and point B of section III of each main cable centerline will be located at the positions prescribed in the shape control problem (see Figure 8.1).

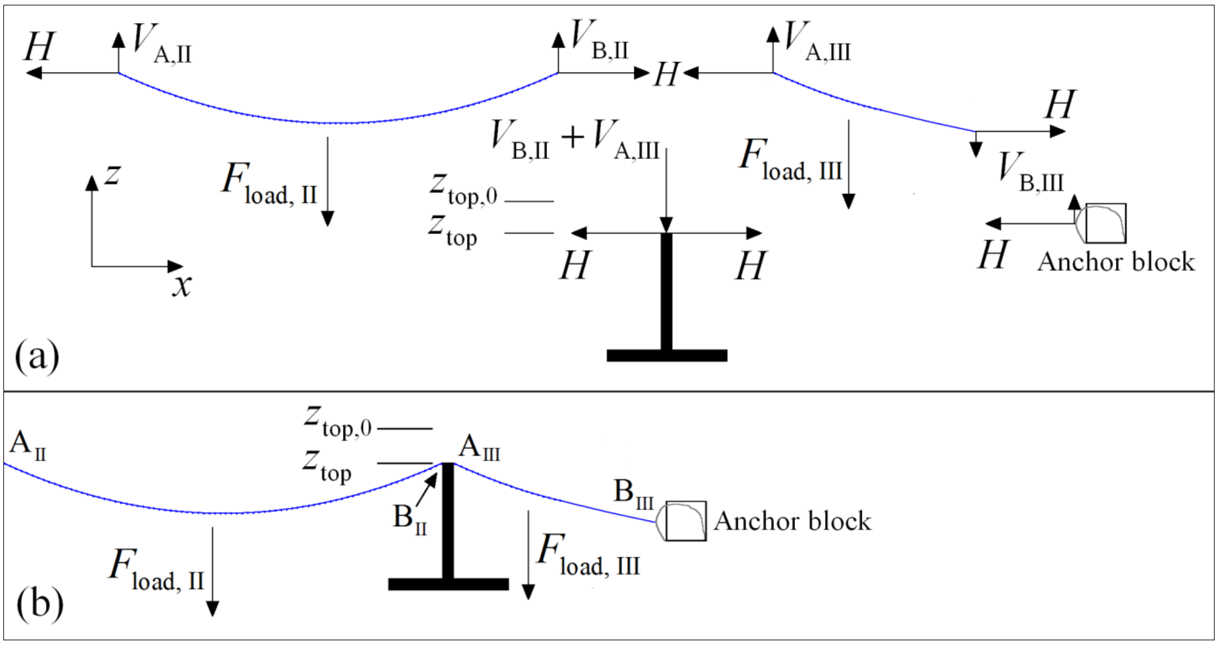


Figure 8.1a: The reaction forces at the ends of the respective main cable sections are obtained from the solution of the shape control problem. Since the reaction forces, acting at the ends of the different sections of the main cables, are known, the deformations of the supports can be calculated.
Figure 8.1b: The dimensions and locations of the supports have been adjusted, in order to ensure that the supports deform in such a way that the end points, of the centerline of each respective main cable section, are located at prescribed positions when the main cables are under the loads assumed in the shape control problem.

In both figures, the deformations of the supports are exaggerated for increased visibility.

9. Conclusions

If the classical solutions by Irvine and Sinclair [2] are applicable, then it is likely that the methods presented in the present paper can be used to control the shape of the cables involved. It turned out that once the functions for the position co-ordinates of the cable centerline have been programmed according to the classical solutions, the shape control problems concerning certain technically important problems can be solved accurately in a straightforward manner, using only a few extra rows of program code. In addition, the numerical solutions are usually obtained in a short period of time. The excellent property of the classical solutions, which enables the solution to be obtained with the external loads applied in full from the start, is retained in the shape control method. Also, since the classical soluions for inextensible cables can be obtained graphically, it is, for the shape control problems, usually easy to obtain the starting values for the numerical solution procedure.

Appendix A

A brief description of the method used for numerically solving nonlinear equations

In the examples of the present paper, we numerically solve single nonlinear scalar equations, as well as systems of nonlinear scalar equations, by using Newton's method. The convergence behaviour of the solution process is shown in some of the examples.

In this appendix, we describe, in brief, the algorithm used, and define the quantities that are relevant for the shown convergence behaviour.

The system of equations to be solved can schematically be written as

$$\begin{aligned} 0 &= f_1(g_1,\ g_2,\ \ldots\ ,\ g_m) \\ 0 &= f_2(g_1,\ g_2,\ \ldots\ ,\ g_m) \\ &\vdots \\ 0 &= f_m(g_1,\ g_2,\ \ldots\ ,\ g_m), \end{aligned} \tag{A.1}$$

where $f_b,\ b = 1, 2,\ \ldots\ , m$, are the real valued functions of the unknown quantities g_b, $b = 1, 2,\ \ldots\ , m$, which the system of equations is solved for. However, in many problems, some of the functions $f_1, f_2,\ \ldots\ , f_m$, are only functions of some of the variables g_b, $b = 1, 2,\ \ldots\ ,$ m.

The Newton iteration scheme is given by

$$\boldsymbol{g}^{(k+1)} = \boldsymbol{g}^{(k)} - \left(\boldsymbol{J_f}\left(\boldsymbol{g}^{(k)}\right)\right)^{-1} \boldsymbol{f}\left(\boldsymbol{g}^{(k)}\right),\ k = 0,\ 1,\ 2,\ \ldots\ , \tag{A.2}$$

where

$$\boldsymbol{g} = \begin{pmatrix} g_1 & g_2 & \ldots & g_m \end{pmatrix}^T,$$

$$\boldsymbol{f}(\boldsymbol{g}) = \begin{pmatrix} f_1(\boldsymbol{g}) & f_2(\boldsymbol{g}) & \ldots & f_m(\boldsymbol{g}) \end{pmatrix}^T,$$

and

$$\boldsymbol{J_f} = \begin{pmatrix} \dfrac{\partial f_1}{\partial g_1} & \dfrac{\partial f_1}{\partial g_2} & \cdots & \dfrac{\partial f_1}{\partial g_m} \\ \dfrac{\partial f_2}{\partial g_1} & \dfrac{\partial f_2}{\partial g_2} & \cdots & \dfrac{\partial f_2}{\partial g_m} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial g_1} & \dfrac{\partial f_m}{\partial g_2} & \cdots & \dfrac{\partial f_m}{\partial g_m} \end{pmatrix} \tag{A.3}$$

is the Jacobian matrix of the function $\boldsymbol{f}(\boldsymbol{g})$.

Newton's method will only converge if the starting values of $g_b,\ b = 1, 2,\ \ldots\ , m,$ contained in the matrix $\boldsymbol{g}^{(k\ =\ 0)}$, are sufficiently close to the correct solution of equations (A.1). If the problem involves determining one or two parameters, the starting values, and the solution, can usually be obtained graphically.

In the present paper, the derivatives in the Jacobian matrix are calculated analytically in Chapters 2 to 5, whereas they are calculated numerically in Chapters 6 and 7.

Numerical calculation of the v-th column of $\boldsymbol{J_f}$, here denoted $(\boldsymbol{J_f})_v,\ v = 1, 2,\ \ldots\ , m,$ is performed by using forward differences according to

$$(\boldsymbol{J_f})_v = \frac{\partial \boldsymbol{f}}{\partial g_v} \approx \frac{\boldsymbol{f}(\boldsymbol{g} + h\boldsymbol{e}_v) - \boldsymbol{f}(\boldsymbol{g})}{h}, \tag{A.4}$$

where h is a constant small number, and $\boldsymbol{e}_v$ is a unit column vector with the entry of the v-th row equal to 1, and 0 otherwise.

In some problems, it may be advantageous if h is not the same for all values of v. If this is the case, Equation (A.4) can be rewritten as

$$(\boldsymbol{J_f})_v = \frac{\partial \boldsymbol{f}}{\partial g_v} \approx \frac{\boldsymbol{f}(\boldsymbol{g} + (\Delta g_v)\boldsymbol{e}_v) - \boldsymbol{f}(\boldsymbol{g})}{\Delta g_v}, \tag{A.5}$$

where $\Delta g_v,\ v = 1, 2,\ \ldots\ , m,$ are small constant numbers.

We consider in this paper convergence of the Newton iteration process to have occurred if

$$\left\| \boldsymbol{f}\left(\boldsymbol{g}^{(k+1)}\right) \right\|_2 \leq \varepsilon_N, \tag{A.6}$$

where ε_N is a small number. In the examples given in this paper, the convergence behaviour of the Newton iteration procedure is described by the development of $\|\ \boldsymbol{f}(g^{(k+1)})\ \|_2$ as a function of the iteration number $i = k + 1,\ k = 0, 1, 2,\ \ldots\ .$

Appendix B

MATLAB-code for selected functions and equations

For convenience, the MATLAB-code for some of the most useful functions and system of equations used in the present paper is given. It is often not possible or convenient to use the same notations when programming as in the text of the theory. Therefore, in some cases other variable names are used in the MATLAB-code than in the theory text. A selection of parameters and their corresponding variable names used in the MATLAB-code are given in table B1.

Table B1

Variable name used in the theory text	Variable name used in the MATLAB-code
s	`s`
H	`H`
V_A	`VA`
L_0	`Lo`
EA_0	`EA`
N_F	`NF`
$\boldsymbol{S_j}$	`Sj`
s_n	`sn`
n	`n`
s_k	`sk`
s_{k-1}	`sk_1`
$x_{cB} = x_B - x_A$	`xcB = xB - xA`
$z_{cB} = z_B - z_A$	`zcB = zB - zA`
$z_{cD} = z_D - z_A$	`zcD = zD - zA`
$x_{cF} = x_F - x_A$	`xcF = xF - xA`
$z_{cF} = z_F - z_A$	`zcF = zF - zA`
s_F	`sF`

In the programme code, we have introduced the matrices `Fj` and `Xcj`, which, respectively, contain the magnitude of every vertical point force, and the values of the prescribed co-ordinates x_{cj}, $j = 1, 2, \ ... \ , N_F$, according to

$$\texttt{Fj} = \begin{pmatrix} F_1 & F_2 & \dots & F_{N_F} \end{pmatrix}^T, \qquad \texttt{Xcj} = \begin{pmatrix} x_{c1} & x_{c2} & \dots & x_{cN_F} \end{pmatrix}^T,$$

where

$$x_{cj} = x_j - x_A, \quad j = 1, 2, \ \dots \ , N_F.$$

B.1 MATLAB-code for f_{x_c} in Equation (6.20)

```
function x_c = f_x_c(s,H,VA,Lo,qc,EA,NF,Sj,Fj)

if (s < 0) || (s > Lo);

    disp('The value of s is invalid')

    disp(['s = ' num2str(s)]);

    return

end

C1 = H/qc;

if NF == 0

    n = 0;

    sn = 0;

    sum_Fj = 0;

end

if NF >= 1

    Sj_Q = [0; Sj; Lo];

    %Determine n and sn if 0 <= s < Lo:

    if s < Lo

        STOP = 0;

        k = 0;

        while STOP == 0

            k = k + 1;

            if (s >= Sj_Q(k,1)) && (s < Sj_Q((k + 1),1))

                STOP = 1;

            end

        end
```

```
        n = k - 1;

        sn = Sj_Q(k,1);

    end

    %Determine n and sn if s = Lo:

    if s == Lo

        n = NF;

        sn = Sj(NF,1);

    end

    sum_Fj = sum(Fj(1:n,1));

end

x_c = (H/EA)*s;

x_c = x_c + C1*asinh((qc*s  - VA + sum_Fj)/H) + ...
    - C1*asinh((qc*sn - VA + sum_Fj)/H);

for j = 1:n

    sk   = Sj_Q((j + 1),1);

    sk_1 = Sj_Q(j,1);

    sum_Fj = sum(Fj(1:(j - 1)));

    x_c = x_c + ...
        + C1*asinh((qc*sk   - VA + sum_Fj)/H) + ...
        - C1*asinh((qc*sk_1 - VA + sum_Fj)/H);

end
```

B.2 MATLAB-code for f_{z_c} in Equation (6.21)

```
function z_c = f_z_c(s,H,VA,Lo,qc,EA,NF,Sj,Fj)

if (s < 0) || (s > Lo);

    disp('The value of s is invalid')

    disp(['s = ' num2str(s)]);
```

```
        return

    end

    C2 = 1/qc;
    C3 = 1/EA;
    H2 = H^2;

    if NF == 0

        n = 0;

        sn = 0;

        sum_Fj = 0;

        sum_Fj_sj = 0;

    end

    if NF >= 1

        Sj_Q = [0; Sj; Lo];

        %Determine n and sn if 0 <= s < Lo:

        if s < Lo

            STOP = 0;

            k = 0;

            while STOP == 0

                k = k + 1;

                if (s >= Sj_Q(k,1)) && (s < Sj_Q((k + 1),1))

                    STOP = 1;

                end

            end

            n = k - 1;

            sn = Sj_Q(k,1);

        end

        %Determine n and sn if s = Lo:
```

```
    if s == Lo

        n = NF;

        sn = Sj(NF,1);

    end

    sum_Fj = sum(Fj(1:n,1));

    sum_Fj_sj = 0;

    for k = 1:n

        sum_Fj_sj = sum_Fj_sj + Fj(k,1)*Sj(k,1);

    end

end

z_c = C3*((qc/2)*s^2 - VA*s);

z_c = z_c + C3*s*sum_Fj;

z_c = z_c - C3*sum_Fj_sj;

z_c = z_c + C2*sqrt(H2 + (qc*s  - VA + sum_Fj)^2) + ...
    - C2*sqrt(H2 + (qc*sn - VA + sum_Fj)^2);

for j = 1:n

    sk   = Sj_Q((j + 1),1);

    sk_1 = Sj_Q(j,1);

    sum_Fj = sum(Fj(1:(j - 1)));

    z_c = z_c + ...
        + C2*sqrt(H2 + (qc*sk   - VA + sum_Fj)^2) + ...
        - C2*sqrt(H2 + (qc*sk_1 - VA + sum_Fj)^2);

end
```

B.3 MATLAB-code for Equations (7.1)

```
function F = For_Equations_7_1(G,qc,EA,NF,Fj,Xcj,xcB,zcB,zcD)

H = G(1,1);

VA = G(2,1);

Lo = G(3,1);

Sj = G(4:(3 + NF));

F = zeros((3 + NF),1);

F(1,1) = -xcB + f_x_c(Lo,H,VA,Lo,qc,EA,NF,Sj,Fj);

F(2,1) = -zcB + f_z_c(Lo,H,VA,Lo,qc,EA,NF,Sj,Fj);

F(3,1) = -zcD + f_z_c(Lo/2,H,VA,Lo,qc,EA,NF,Sj,Fj);

for j = 1:NF

    F((j + 3),1) = -Xcj(j,1) + f_x_c(Sj(j,1),H,VA,Lo,qc,EA,NF,Sj,Fj);

end
```

B.4 MATLAB-code for Equations (7.2)

```
function F = For_Equations_7_2(G,qc,EA,NF,Fj,Xcj,xcB,zcB,zcD)

H = G(1,1);

VA = G(2,1);

Lo = G(3,1);

Sj(1:NF/2,1) = G(4:(3 + NF/2));

a = NF;

c = 1;

for b = (NF/2 + 1):NF

    Sj(a,1) = Lo - Sj(c,1);

    a = a - 1;

    c = c + 1;

end

F = zeros((3 + (NF/2)),1);

F(1,1) = -xcB + f_x_c(Lo,H,VA,Lo,qc,EA,NF,Sj,Fj);

F(2,1) = -zcB + f_z_c(Lo,H,VA,Lo,qc,EA,NF,Sj,Fj);

F(3,1) = -zcD + f_z_c(Lo/2,H,VA,Lo,qc,EA,NF,Sj,Fj);

for j = 1:(NF/2)

   F((j + 3),1) = -Xcj(j,1) + f_x_c(Sj(j,1),H,VA,Lo,qc,EA,NF,Sj,Fj);

end
```

B.5 MATLAB-code for Equations (7.4)

```
function F = For_Equations_7_4(G,H,qc,EA,NF,Fj,Xcj,xcF,zcF,xcB)

VA = G(1,1);

Lo = G(2,1);

sF = G(3,1);

Sj = G(4:(NF + 3),1);

F = zeros((3 + NF),1);

F(1,1) = -xcF + f_x_c(sF,H,VA,Lo,qc,EA,NF,Sj,Fj);

F(2,1) = -zcF + f_z_c(sF,H,VA,Lo,qc,EA,NF,Sj,Fj);

F(3,1) = -xcB + f_x_c(Lo,H,VA,Lo,qc,EA,NF,Sj,Fj);

for j = 1:NF

    F((j + 3),1) = -Xcj(j,1) + f_x_c(Sj(j,1),H,VA,Lo,qc,EA,NF,Sj,Fj);

end
```

Appendix C

Numerical values of the matrix $\boldsymbol{S_j}$ of selected examples

The matrix $\boldsymbol{S_j}$ of every problem is a $(N_F \times 1)$ matrix, but in order to reduce the number of pages used, the $\boldsymbol{S_j}$ matrix of each problem is here divided and written in separate columns. The values of the matrices are copied directly from MATLAB, and all numerical values have the metric unit: meter.

$\boldsymbol{S_j}$ of Example 3.1, III

```
Sj = 1000*[

0.008935681203518
0.026733765178290
0.044454022629818
0.062098282989069
0.079668388698317
0.097166195113882
0.114593570394737
0.131952395376751
0.149244563432287
0.166471980314978
0.183636563989461
0.200740244445938
0.217784963499430
0.234772674573638
0.251705342469358
0.268584943117453
0.285413463316380
0.302192900454370
0.318925262216352
0.335612566275786
0.352256839971595
0.368860119970450
0.385424451914688
0.401951890056198
0.418444496876664
0.434904342694582
0.451333505259530
0.467734069334197
0.484108126264736
0.500457773540047
0.516785114340611
0.533092257077586
0.549381314922843
0.565654405330733
0.581913649552335
0.598161172143012
0.614399100464113
0.630629564179681
0.646854694749037
0.663076624916157
0.679297488196735
0.695519418363855
0.711744548933211
0.727975012648779
0.744212940969880
0.760460463560557
0.776719707782158
0.792992798190048
0.809281856035306
0.825588998772281
0.841916339572845
0.858265986848156
0.874640043778695
0.891040607853361
0.907469770418310
0.923929616236228
0.940422223056693
0.956949661198203
0.973513993142442
0.990117273141297
1.006761546837106
1.023448850896540
1.040181212658522
1.056960649796512
1.073789169995439
1.090668770643534
1.107601438539254
1.124589149613462
1.141633868666954
1.158737549123431
1.175902132797914
1.193129549680605
1.210421717736141
1.227780542718155
1.245207917999010
1.262705724414575
1.280275830123823
1.297920090483074
1.315640347934602
1.333438431909374]
```

S_j of Example 3.2, II

```
Sj = 100*[

 0.087735389593507
 0.262459024076363
 0.436385163867106
 0.609529424173144
 0.781907533116952
 0.953535331471964
 1.124428772300687
 1.294603920492249
 1.464076952196783
 1.632864154154295
 1.800981922915441
 1.968446763952162
 2.135275290656163
 2.301484223223204
 2.467090387421724
 2.632110713244208
 2.796562233440142
 2.960462081929427
 3.123827492095653
 3.286675794958565
 3.449024417225592
 3.610890879222452
 3.772292792703130
 3.933247858539984
 4.093773864294759]
```

S_j of Example 3.3, II

```
Sj = 1000*[

 0.008386426970389   0.346179729541085   0.672630279925836   1.012527487664198
 0.025085535760006   0.361786578177018   0.688316849717900   1.029455014145318
 0.041707631537095   0.377370085778981   0.704036933002173   1.046468448006129
 0.058255101430792   0.392932947924757   0.719793180132514   1.063570078806020
 0.074730352226870   0.408477867164917   0.735588229745490   1.080762175326593
 0.091135810071213   0.424007552139710   0.751424708022774   1.098046985475696
 0.107473920141544   0.439524716681539   0.767305227980677   1.115426736217916
 0.123747146287037   0.455032078905068   0.783232388787928   1.132903633530631
 0.139957970635587   0.470532360287072   0.799208775112673   1.150479862384670
 0.156108893168577   0.486028284738201   0.815236956499572   1.168157586748607
 0.172202431263115   0.501522577668880   0.831319486777697   1.185938949615697
 0.188241119201799   0.517017965051570   0.847458903499844   1.203826073052455]
 0.204227507650231   0.532517172481683   0.863657727413714
 0.220164163102567   0.548022924239400   0.879918461965315
 0.236053667295567   0.563537942354670   0.896243592834777
 0.251898616591706   0.579064945677649   0.912635587504720
 0.267701621332039   0.594606648956779   0.929096894861124
 0.283465305159667   0.610165761926708   0.945629944826591
 0.299192304314742   0.625744988408162   0.962237148025761
 0.314885266902125   0.641347025421838   0.978920895482565
 0.330546852132900   0.656974562318310   0.995683558348902
```

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