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MATEMATIK

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SYSTEM OCH TRANSFORMER
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1.a) The system is linear, time-invariant and causal.

b) The system is **not** linear, time-invariant and causal.

c) The system is linear, **not** time-invariant and **not** causal.

2.a) The characteristic polynomial of A is $p_A(\lambda) = \lambda^2 + 10\lambda + 24$. We get the eigenvalues $\lambda_1 = -4$ and $\lambda_2 = -6$ with corresponding eigenvectors $s_1 = c_1(1,1)^T$ and $s_2 = c_2(-1,1)^T$, $c_i \neq 0, i = 1,2$. With

$$S = A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

we get

$$\begin{aligned} A^n &= S \begin{bmatrix} -4 & 0 \\ 0 & -6 \end{bmatrix}^n S^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} (-4)^n & 0 \\ 0 & (-6)^n \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} (-1)^n(4^n + 6^n) & (-1)^n(4^n - 6^n) \\ (-1)^n(4^n - 6^n) & (-1)^n(4^n + 6^n) \end{bmatrix} \end{aligned}$$

b) The time-discrete system has the solution

$$x(k) = A^k x(0) = \frac{1}{2} \begin{bmatrix} (-1)^k(4^k + 6^k) & (-1)^k(4^k - 6^k) \\ (-1)^k(4^k - 6^k) & (-1)^k(4^k + 6^k) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (-1)^k 4^k \\ (-1)^k 4^k \end{bmatrix}$$

c) With the given initial conditions we get

$$\frac{x_0(k)}{x_1(k)} = 1.$$

For general initial conditions we have the general solution

$$x(k) = c_1(-4)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2(-6)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and hence,

$$\lim_{k\to\infty} \frac{x_0(k)}{x_1(k)} = -1.$$

3.a) We will use the one-sided Laplacetransform. Then

$$\begin{aligned} \mathcal{L}\left(e^{At}\theta(t)\right)(s) = (sI - A)^{-1} &= \begin{bmatrix} s & 0 & 1 \\ 0 & s-1 & 0 \\ -1 & 0 & s \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \frac{s}{s^2+1} & 0 & -\frac{1}{s^2+1} \\ 0 & \frac{1}{s-1} & 0 \\ \frac{1}{s^2+1} & 0 & \frac{s}{s^2+1} \end{bmatrix}. \end{aligned}$$

The inverse transform and the idendity theorem for analytic functions gives

$$e^{At} = \begin{bmatrix} \cos t & 0 & -\sin t \\ 0 & e^t & 0 \\ \sin t & 0 & \cos t \end{bmatrix}.$$

b) Since i is an eigenvalue for A we cannot use the method of a generalized stationary solution. Instead we will use the (integrating factor) formula

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} f(\tau)\, d\tau.$$

That gives

$$\begin{aligned} x(t) &= \begin{bmatrix} \cos t & 0 & -\sin t \\ 0 & e^t & 0 \\ \sin t & 0 & \cos t \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \int_0^t \begin{bmatrix} \cos(t-\tau) & 0 & -\sin(t-\tau) \\ 0 & e^{t-\tau} & 0 \\ \sin(t-\tau) & 0 & \cos(t-\tau) \end{bmatrix} \cdot \begin{bmatrix} \cos t \\ 0 \\ 0 \end{bmatrix} d\tau \\ &= \begin{bmatrix} \cos t - \sin t \\ e^t \\ \sin t + \cos t \end{bmatrix} + \int_0^t \begin{bmatrix} \cos(t-\tau)\cos\tau \\ 0 \\ \sin(t-\tau)\cos\tau \end{bmatrix} d\tau = \begin{bmatrix} \cos t - \sin t \\ e^t \\ \sin t + \cos t \end{bmatrix} + \begin{bmatrix} \frac{1}{2}t\cos t + \frac{1}{2}\sin t \\ 0 \\ \frac{1}{2}t \sin t \end{bmatrix} \\ &= \begin{bmatrix} \cos t + \frac{1}{2}t\cos t - \frac{1}{2}\sin t \\ e^t \\ \sin t + \frac{1}{2}t\sin t + \cos t \end{bmatrix}. \end{aligned}$$

4. Since the system is causal we will determine the transfer function by means of causal insignals. Then $w(t) = w(t)\theta(t)$ and $y(t) = y(t)\theta(t)$. The defining equation can now be written as

$$y^{(3)} - y'' + 2y = 2w * \theta + w' - 2w.$$

The Laplace transform gives with $Y = \mathcal{L}(y)$ and $W = \mathcal{L}(w)$

$$s^3Y(s) - s^2Y(s) + 2Y(s) = 2W(s)\frac{1}{s} + sW(s) - 2W(s).$$

Hence, the transfer function is

$$H(s) = \frac{Y(s)}{W(s)} = \frac{s^2 - 2s + 2}{s(s^3 - s^2 + 2)} = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}.$$

The inverse transform gives the impulse response

$$h(t) = \theta(t) - e^{-t}\theta(t).$$

The stepfunction response is

$$S(\theta)(t) = \int_{-\infty}^t h(t)\, dt = (t + e^{-t} - 1)\theta(t).$$

The system is unstable since

$$\int_{-\infty}^{\infty} |h(t)|\, dt = +\infty.$$

5. We will use the one-sided Laplacetransform. The transformed equation is

$$s^2Y(s) - 4 + 2sY(s) + 10Y(s) = 2\frac{s}{s^2+1} + 9\frac{1}{s^2+1}$$

where $Y = \mathcal{L}(y\theta)$. Therefore

$$Y(s) = \frac{4s^2+2s+13}{(s^2+1)((s+1)^2+9)} = \frac{1}{s^2+1} + \frac{3}{(s+1)^2+3^2}.$$

The inverse transformation gives

$$y(t) = \sin t + e^{-t}\sin(3t).$$

Checking the initial conditions we get $y(0) = 0$ and

$$y'(0) = \cos t - e^{-t}\sin(3t) + e^{-t}3\cos(3t)|_{t=0} = 4.$$

6.a) A simple system is given by $y(t) = S(w)(t) = w'(t)$. It has transfer function $H(s) = s, s \in \mathbb{C}$ but for $a = 1$ we have

$$S(\cos t) = -\sin t \neq i\cos t = H(i)\cos t.$$

b) Under the conditions we have that $H(i\omega)$ is the frequency function. The table method gives the Fourier transform

$$\mathcal{F}(\cos(at))(\omega) = \pi(\delta(\omega - a) + \delta(\omega + a)).$$

Therefore the outsignal is

$$\begin{aligned} S(\cos(a\tau))(t) &= (\mathcal{F}^{-1}(\pi(\delta(\omega-a) + \delta(\omega+a))H(i\omega)) \\ &= \mathcal{F}^{-1}(\pi H(ia)\delta(\omega-a) + \pi H(-ia)\delta(\omega+a)) \\ &= \mathcal{F}^{-1}(\pi H(ia)\delta(\omega-a) + \pi H(ia)\delta(\omega+a)) \\ &= \mathcal{F}^{-1}(\pi H(ia)(\delta(\omega-a) + \delta(\omega+a))) \\ &= H(ia)\cos(at). \end{aligned}$$

c) By using Parseval's formula twice we have

$$\begin{aligned} \int_{-\infty}^{\infty} |f(t)|^2\,dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{1-i\omega}{1+i\omega}\right|^2 \frac{\sin^2\omega}{w^2}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin^2\omega}{\omega^2}\,d\omega \\ &= \int_{-\infty}^{\infty} \left(\frac{1}{2}(\theta(t+1) - \theta(t-1))\right)^2\,dt \\ &= \frac{1}{4}\int_{-1}^{1} dt = \frac{1}{2}. \end{aligned}$$