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MATEMATIK

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1. By Laplacetransforming we get

$$\frac{Y}{W} = \frac{s^2+4s+4}{s^3+2s^2+2s} = \frac{s^2+2s+2}{s(s^2+2s+2)} + \frac{2s+2}{s(s^2+2s+2)} = \frac{1}{s} + \frac{1}{s} - \frac{s}{s^2+2s+2}$$
$$= \frac{2}{s} - \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}$$

where $Y = \mathcal{L}y$ och $W = \mathcal{L}w$. We also have for the transfer function H that $H = \dfrac{Y}{W}$. The causal inverstransform gives the impulse response

$$h(t) = \big(2 - e^{-t}(\cos t - \sin t)\big)\,\theta(t).$$

Since $S(\delta') = S'(\delta) = h'(t)$ we get

$$S(\delta')(t) = 2e^{-t}\cos t \cdot \theta(t) + \delta(t).$$

We have that $H(s)$ has a single pole at 0 and all other in the left halfplane. Therefore the system is not stable (It is marginal stable).

2. a) The charcteristic polynomial of A is $p_A(\lambda) = \lambda^2 - 5\lambda + 4$. That gives the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 4$ with corresponding (particular) eigenvectors $s_1 = (-1,1)^T$ and $s_2 = (1,2)^T$. Hence with

$$S = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$$

and

$$S^{-1} = \frac{1}{3}\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$$

we have

$$e^{tA} = S\begin{bmatrix} e^t & 0 \\ 0 & e^{4t} \end{bmatrix} S^{-1} = \frac{1}{3}\begin{bmatrix} 2e^t + e^{4t} & e^{4t} - e^t \\ 2e^{4t} - 2e^t & 2e^{4t} + e^t \end{bmatrix}.$$

b) Similarily to a) we compute

$$A^n = S\begin{bmatrix} 1 & 0 \\ 0 & 4^n \end{bmatrix} S^{-1} = \frac{1}{3}\begin{bmatrix} 2+4^n & 4^n - 1 \\ 2\cdot 4^n - 2 & 2\cdot 4^n + 1 \end{bmatrix}.$$

and finally

$$\begin{bmatrix} x_k \\ y_k \end{bmatrix} = A^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2+4^k \\ 2\cdot 4^k - 2 \end{bmatrix}.$$

c) There is no such matrix B since the exponential of a 3×3 matrix cannot contain a term t^3.

3. We choose to Laplacetransform the system

$$s^2X(s) - 1 = -sY(s)$$
$$s^2Y(s) = sX(s)$$

where $X(s) = \mathcal{L}_I(x(t))(s), Y(s) = \mathcal{L}_I(y(t))(s)$. Solving for $X(s)$ we get $X(s) = \dfrac{1}{1+s^2}$. The inverse Laplacetransform yields

$$x(t) = \sin t, \qquad t \geq 0.$$

The second (original) equation gives $y''(t) = \cos t$ and $y(t) = -\cos t + Ct + D$. The initial conditions for y give

$$y(t) = -\cos t + 1, \qquad t \geq 0.$$

4. a) A function $g\colon \mathbb{R} \to \mathbb{C}$ is real if and only if $\overline{g(t)} = g(t)$ for all $t \in \mathbb{R}$. Hence, $\overline{f(t)} = f(t)$ and we need to show that $\overline{\mathcal{F}(f)(\omega)} = \mathcal{F}(f)(\omega)$:

$$\begin{aligned}
\overline{\mathcal{F}(f)(\omega)} &= \overline{\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt} = \int_{-\infty}^{\infty} \overline{f(t)e^{-i\omega t}}\,dt \\
&= \int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt = \int_{-\infty}^{\infty} f(-t)e^{-i\omega t}\,dt = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt \\
&= \mathcal{F}(f)(\omega).
\end{aligned}$$

b) We have

$$\begin{aligned}
\mathcal{F}(f)(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt = \int_{-1}^{1} (1-|t|)e^{-i\omega t}\,dt \\
&= \int_{0}^{1} (1-t)e^{-i\omega t}\,dt + \int_{-1}^{0} (1+t)e^{-i\omega t}\,dt \\
&= \frac{(2e^{i\omega} - e^{2i\omega} - 1)e^{-i\omega}}{\omega^2} = \frac{2-2\cos(\omega)}{\omega^2} \\
&= 4\frac{\sin^2\left(\frac{\omega}{2}\right)}{w^2}.
\end{aligned}$$

c) We first remark (by the table method) that

$$\mathcal{F}^{-1}\left(\frac{\sin^2\omega}{\omega^2}\right)(t) = \frac{1}{4}(2-|t|)(\theta(t+2) - \theta(t-2)).$$

. We will use Parseval's formula:

$$\begin{aligned}
\int_0^{\infty} \frac{\sin^2\omega}{\omega^2+\omega^4}\,d\omega &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin^2\omega}{\omega^2}\cdot\frac{1}{(1+\omega^2)}\,d\omega \\
&= \pi\int_{-\infty}^{\infty} \mathcal{F}^{-1}\left(\frac{\sin^2\omega}{\omega^2}\right)(t)\cdot\mathcal{F}^{-1}\left(\frac{1}{1+\omega^2}\right)(t)\,dt \\
&= \pi\int_{-2}^{2} \frac{1}{4}(2-|t|)\cdot\frac{1}{2}e^{-|t|}\,dt = \frac{\pi}{4}\int_0^2 (2-t)e^{-t}\,dt \\
&= \frac{\pi}{4}\left(1+\frac{1}{e^2}\right).
\end{aligned}$$

5. a) We use the definition of the Laplacetransform for $\mathbb{R}e(s) > 0$ and substitute in the following $u = st$ and then $w = \sqrt{u}$:

$$\mathcal{L}_I\left(\frac{1}{\sqrt{t}}\right)(s) = \int_0^\infty e^{-st}\frac{1}{\sqrt{t}}\,dt = \frac{1}{\sqrt{s}}\int_0^\infty e^{-u}\frac{1}{\sqrt{u}}\,du$$
$$= \frac{2}{\sqrt{s}}\int_0^\infty e^{-w^2}\,dw = \frac{\sqrt{\pi}}{\sqrt{s}}.$$

Note that we have to fix a branch of $\sqrt{s}$. Further we use (induction) that

$$\mathcal{L}_I\left(\frac{t^n}{\sqrt{t}}\right)(s) = -\frac{d}{ds}\mathcal{L}_I\left(\frac{t^{n-1}}{\sqrt{t}}\right)(s).$$

We get for $n > 0$

$$\mathcal{L}_I\left(\frac{t^n}{\sqrt{t}}\right)(s) = \prod_{k=1}^{n}\left(k - \frac{1}{2}\right)\sqrt{\pi}\cdot\frac{1}{s^{n+\frac{1}{2}}} = \frac{(2n)!}{n!4^n}\sqrt{\pi}\cdot\frac{1}{s^{n+\frac{1}{2}}}.$$

In particular we have

$$\mathcal{L}_I\left(\frac{t}{\sqrt{t}}\right)(s) = \frac{\sqrt{\pi}}{2}\frac{1}{s^{1+\frac{1}{2}}}, \qquad \mathcal{L}_I\left(\frac{t^2}{\sqrt{t}}\right)(s) = \frac{3\sqrt{\pi}}{4}\frac{1}{s^{2+\frac{1}{2}}}.$$

b) We observe that after multipying with $\theta(t)$ the equation becomes a convolution equation.

$$\theta(t) + t\theta(t) + t^2\theta(t) = \left(\frac{1}{\sqrt{t}}\theta(t)\right) * \left(f(t)\theta(t)\right).$$

Laplacetransforming yields

$$\frac{1}{s} + \frac{1}{s^2} + \frac{2}{s^3} = \frac{\sqrt{\pi}}{\sqrt{s}}\mathcal{L}(f\theta)(s)$$

or

$$\mathcal{L}(f\theta)(s) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{s^{\frac{1}{2}}} + \frac{1}{s^{1+\frac{1}{2}}} + 2\frac{1}{s^{2+\frac{1}{2}}}\right).$$

Using a) we can inversetransform

$$f(t) = \frac{1}{\pi}\left(\frac{1}{\sqrt{t}} + \frac{2t}{\sqrt{t}} + \frac{8t^2}{3\sqrt{t}}\right).$$

6. a) Let us consider the (non identically zero) function $f(t) = (1 - |t|)(\theta(t+1) - \theta(t-1))$. Then

$$\int_0^1 g(t)\cos(\lambda t)\,dt = \frac{1}{2}\int_{-\infty}^\infty f(t)\cos(\lambda t)\,dt = \frac{1}{2}\mathbb{R}e\left(\mathcal{F}(f)(\lambda)\right)$$
$$= \frac{1}{2}\mathbb{R}e\left(4\frac{\sin^2\left(\frac{\lambda}{2}\right)}{\lambda^2}\right) = 2\frac{\sin^2\left(\frac{\lambda}{2}\right)}{\lambda^2} \geq 0$$

for all $\lambda \in \mathbb{R}$.

b) Let f be a function that satisfies the conditions in a). We denote by D the distributional derivative and by ' the ordinary derivative. Then

$$\int_0^1 f(t)\cos(\lambda t)\,dt = \int_{-\infty}^{\infty} f(t)(\theta(t)-\theta(t-1))\cos(\lambda t)\,dt$$

Noting that $f(t)(\theta(t)-\theta(t-1)) = 0$ for $|t| > 1$ and integrating by parts (in the distributional sense) we get

$$\begin{aligned}\int_0^1 f(t)\cos(\lambda t)\,dt &= -\frac{1}{\lambda}\int_{-\infty}^{\infty} \sin(\lambda t) D\Big(f(t)(\theta(t)-\theta(t-1))\Big)\,dt \\ &= -\frac{1}{\lambda}\int_{-\infty}^{\infty}\Big(f'(t)(\theta(t)-\theta(t-1))\sin(\lambda t) + f(t)\delta(t)\sin(\lambda t) - f(t)\delta(t-1)\sin(\lambda t)\Big)\,dt \\ &= -\frac{1}{\lambda}\left(\int_{-\infty}^{\infty} f'(t)(\theta(t)-\theta(t-1))\sin(\lambda t)\,dt - f(1)\sin\lambda\right).\end{aligned}$$

Since $f \in C^1([0,1])$ by the Riemann-Lebesgue Lemma

$$\int_{-\infty}^{\infty} f'(t)(\theta(t)-\theta(t-1))\sin(\lambda t)\,dt = \mathbb{I}m(\mathcal{F}(f'(t)(\theta(t)-\theta(t-1))(\lambda)) \to 0$$

as $|\lambda| \to \infty$. Hence if $f(1) \neq 0$ for large $n \in \mathbb{N}$ we have

$$\operatorname{sign}\left(\int_0^1 f(t)\cos\left(n\frac{\pi}{2}t\right)\,dt\right) = \operatorname{sign}\left(f(1)\sin\left(n\frac{\pi}{2}\right)\right) = \pm 1$$

and the integral cannot be non-negative for all $\lambda \in \mathbb{R}$.

c) Similar as before we have that

$$\int_1^2 f(t)\cos(\lambda t)\,dt = \mathbb{R}e(\mathcal{F}(f(t)(\theta(t-1)-\theta(t-2)))(\lambda).$$

By assumption the real part of the above Fouriertransform is strictly positive for each $\lambda \in \mathbb{R}$. Hence,

$$\int_{-\infty}^{\infty} \mathbb{R}e\left(\mathcal{F}(f(t)(\theta(t-1)-\theta(t-2)))(\lambda)\right)\,d\lambda > 0.$$

The Fourier inversion formula gives

$$0 = f(t)(\theta(t-1)-\theta(t-2))|_{t=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}(f(t)(\theta(t-1)-\theta(t-2)))(\lambda)\,d\lambda \tag{1}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbb{R}e\left(\mathcal{F}(f(t)(\theta(t-1)-\theta(t-2)))(\lambda)\right)\,d\lambda + \tag{2}$$

$$+ i\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbb{I}m\left(\mathcal{F}(f(t)(\theta(t-1)-\theta(t-2)))(\lambda)\right)\,d\lambda \neq 0 \tag{3}$$

A contradiction.

Remark: We can change the condition in the problem to

$$\int_1^2 f(t)\cos(\lambda t)\,dt \geq 0, \quad \lambda \in \mathbb{R}$$

and f is not identically 0. Then the previous arguments read as:

$$\int_{-\infty}^{\infty} \mathbb{R}e\left(\mathcal{F}(f(t)(\theta(t-1)-\theta(t-2)))(\lambda)\right)\, d\lambda \geq 0.$$

The Fourier inversion formula gives

$$0 = f(t)(\theta(t-1)-\theta(t-2))|_{t=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}(f(t)(\theta(t-1)-\theta(t-2)))(\lambda)\, d\lambda \tag{4}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbb{R}e\left(\mathcal{F}(f(t)(\theta(t-1)-\theta(t-2)))(\lambda)\right)\, d\lambda + \tag{5}$$

$$+\, i\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbb{I}m\left(\mathcal{F}(f(t)(\theta(t-1)-\theta(t-2)))(\lambda)\right)\, d\lambda. \tag{6}$$

This implies that

$$\int_{-\infty}^{\infty} \mathbb{R}e\left(\mathcal{F}(f(t)(\theta(t-1)-\theta(t-2)))(\lambda)\right)\, d\lambda = 0$$

and hence,

$$\mathbb{R}e(\mathcal{F}(f(t)(\theta(t-1)-\theta(t-2)))(\lambda)) = \int_{1}^{2} f(t)\cos(\lambda t)\, dt = 0$$

for all $\lambda \in \mathbb{R}$ since the only non-negative piecewise continuous function with zero integral is the zero function. Hence, the Fouriertransform of $f(t)(\theta(t-1)-\theta(t-2))$ is purely imaginary. This implies that $f(t)(\theta(t-1)-\theta(t-2))$ must be an odd function. For if g is a real function with purely imaginary Fouriertransform then

$$g(-t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \sin(\omega(-t))\mathcal{F}(g)(\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \sin(\omega t)\mathcal{F}(g)(\omega)\, d\omega = -g(t).$$

But this means that $f(t)(\theta(t-1)-\theta(t-2)) = 0$ for all $t \in \mathbb{R}$ since it is 0 for all negative t. Consequently, $f(t) = 0$ for all $t \in [1,2]$. A contradiction.