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MATEMATIK

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1.a) We first compute the eigenvalues and (some) eigenvectors of A:

$$\lambda_1 = -1, \qquad s_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \lambda_2 = 8, \qquad s_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Since we have two linearly independent eigenvectors the general solution is then given by

$$\begin{bmatrix} x_k \\ y_k \end{bmatrix} = c_1(-1)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 8^k \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

b) To get a bounded solution we must have $c_2 = 0$. Setting $k = 0$ (and $c_2 = 0$) we get

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \text{or} \qquad x_0 + y_0 = 0.$$

c) With

$$D = S^{-1}AS = \begin{bmatrix} -1 & 0 \\ 0 & 8 \end{bmatrix} = S^{-1}B^3S = \left(S^{-1}BS\right)^3$$

we get (for the real solution)

$$S^{-1}BS = \begin{bmatrix} -\sqrt[3]{1} & 0 \\ 0 & \sqrt[3]{8} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Finally

$$B = S \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} S^{-1} = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}.$$

2. Since $t > 0$ we multiply the equation with $\theta(t)$ and get a convolution equation that holds for all $t \in \mathbb{R}$:

$$3 \cdot (\theta * y\theta)(t) - ty(t)\theta(t) = t^2\theta(t).$$

The Laplacetransform of this equation gives (with $Y(s) = \mathcal{L}(y\theta)(s)$)

$$3\frac{Y(s)}{s} + Y'(s) = \frac{2}{s^3}.$$

The integrating factor is $e^{3\ln s} = s^3$ and therefore

$$\left(s^3Y(s)\right)' = 2 \qquad \text{and hence,} \qquad Y(s) = \frac{2}{s^2} + \frac{C}{s^3}.$$

The inverse Laplacetransform gives

$$y(t)\theta(t) = 2t\theta(t) + \frac{C}{2}t^2\theta(t) \qquad \text{or} \qquad y(t) = 2t + \frac{C}{2}t^2, t > 0.$$

The condition $y(1) = 3$ leads to $C = 2$ and

$$y(t) = 2t + t^2, \quad t > 0.$$

3. Simplifying the right-hand-side we get

$$\left(e^t y'\right)' = t^4 \cos^2(2\pi t)\delta(t-1) = 1^4 \cos^2(2\pi \cdot 1)\delta(t-1) = \delta(t-1).$$

Now we can repeatedly integrate

$$e^t y'(t) = \theta(t-1) + A, \qquad y'(t) = e^{-t}\theta(t-1) + Ae^{-t}$$

and

$$y(t) = \left(-e^{-t} + e^{-1}\right)\theta(t-1) + \tilde{A}e^{-t} + B.$$

The boundary condition $\lim_{t\to-\infty} y(t) = 1$ at $-\infty$ gives $\tilde{A} = 0$ and $B = 1$. The solution is

$$y(t) = \left(e^{-1} - e^{-t}\right)\theta(t-1) + 1.$$

4.a) We compute the particular solution with the complexified right-hand-side

$$\mathbf{f}e^{2it} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot e^{2it}.$$

since $\cos(t) = \mathbb{R}e(e^{2it})$. Then we get

$$x = R_A(2i)\mathbf{f} = \frac{1}{4+12i}\begin{bmatrix} 3+2i & 1 \\ 1 & 3+2i \end{bmatrix}\mathbf{f} \cdot e^{2it} = \frac{1}{40}\begin{bmatrix} 19-17i \\ 11-13i \end{bmatrix} \cdot e^{2it}.$$

Taking the real part we get

$$\begin{cases} x_1(t) = \frac{1}{40}(19\cos(2t) + 17\sin(2t)) \\ x_2(t) = \frac{1}{40}(11\cos(2t) + 13\sin(2t). \end{cases}$$

b) The eigenvalues and eigenvectors of the system matrix are given by

$$\begin{cases} \lambda_1 = -2, \ s_1 = (1,1)^T \\ \lambda_2 = -4, \ s_2 = (-1,1)^T. \end{cases}$$

Since we have two distinct eigenvalues the corresponding eigenvectors are linearly independent and therefore form a base. That gives

$$\begin{cases} x_1(t) = c_1 e^{-2t} - c_2 e^{-4t} + \frac{1}{40}(19\cos(2t) + 17\sin(2t)) \\ x_2(t) = c_1 e^{-2t} + c_2 e^{-4t} + \frac{1}{40}(11\cos(2t) + 13\sin(2t)). \end{cases}$$

The initial values help to find the constants c_1, c_2:

$$\begin{cases} x_1(t) = -\frac{3}{8}e^{-2t} - \frac{1}{10}e^{-4t} + \frac{1}{40}(19\cos(2t) + 17\sin(2t)) \\ x_2(t) = -\frac{3}{8}e^{-2t} + \frac{1}{10}e^{-4t} + \frac{1}{40}(11\cos(2t) + 13\sin(2t)). \end{cases}$$

c) No. The equationen $(s \cdot I - A)x = \mathbf{f}$ has no solution since $s = -2$ is an eigenvalue for the system matrix A and the vector $(1,3)^T$ is not in the image of $(A + 2I)$.
d) We make the ansatz

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \cdot te^{-2t} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \cdot e^{-2t}$$

for a particular solution. We get $A_1 = 2, A_2 = 2, B_1 = 0, B_2 = 1$ and

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} (2t + c_1)e^{-2t} - c_2 e^{-4t} \\ (2t + 1 + c_1)e^{-2t} + c_2 e^{-4t} \end{bmatrix}.$$

5. After simplification we get the equation

$$-\frac{\hbar^2}{2m}\psi''(x) - g\psi(0)\delta(x) = E\psi(x).$$

Fouriertransforming it we arrive at

$$\frac{\hbar^2}{2m}\omega^2\mathcal{F}(\psi)(\omega) - g\psi(0) = E\mathcal{F}(\psi)(\omega)$$

or

$$\mathcal{F}(\psi)(\omega) = \frac{2mg\psi(0)}{\hbar^2}\frac{1}{\omega^2 + \left(-\frac{2mE}{\hbar^2}\right)}.$$

With $\alpha = \sqrt{-\frac{2mE}{\hbar^2}}$ (note $E < 0$!) we get

$$\begin{aligned} \psi(x) &= \frac{2mg\psi(0)}{\hbar^2}\mathcal{F}^{-1}\left(\frac{1}{\omega^2 + \left(-\frac{2mE}{\hbar^2}\right)}\right)(x) \\ &= \frac{2mg\psi(0)}{\hbar^2}\frac{1}{2\alpha}\mathcal{F}^{-1}\left(\frac{1}{\alpha}\frac{2}{\left(\frac{\omega}{\alpha}\right)^2 + 1}\right)(x) \\ &= \frac{mg\psi(0)}{\hbar^2\alpha}e^{-\alpha|x|}. \end{aligned}$$

Setting $x = 0$ gives

$$1 = \frac{mg}{\hbar^2\alpha} \qquad \Longrightarrow \qquad \alpha^2 = \frac{m^2g^2}{\hbar^4}$$

that $\psi(x) = \psi(0)e^{-\alpha|x|}$ and $E = -\frac{mg^2}{2\hbar^2}$.
Moreover,

$$1 = \int_{-\infty}^{\infty}\psi(x)^2\,dx = \psi(0)^2 \cdot 2\int_0^{\infty} e^{-2\alpha|x|}\,dx = \frac{\psi(0)^2}{\alpha}.$$

Hence, $\psi(0) = \sqrt{\alpha}$ and

$$\psi(x) = \frac{\sqrt{mg}}{\hbar}e^{-\frac{mg}{\hbar^2}|x|}.$$

6.a) We have

$$\sin(|t|) = \begin{cases} \sin t & t > 0 \\ -\sin t & t < 0 \end{cases} = \sin t \cdot (2\theta(t) - 1) = \sin t \cdot (\theta(t) - 1) + \sin t \cdot \theta(t).$$

b) The definition strip for the Laplacetransform of $\sin t \cdot (\theta(t) - 1)$ is $\mathbb{R}e(s) < 0$ and for $\sin t \cdot \theta(t)$ is $\mathbb{R}e(s) > 0$. Hence the common definition strip is empty and f has no Laplacetransform in any (open) strip. It is also easy to see that for $\mathbb{R}e(s) = 0$ the Laplacetransform diverges.

c) To compute the Fouriertransform of f we start with

$$\begin{aligned}\mathcal{F}\left(2\theta(t)-1\right)(\omega) &= 2\mathcal{F}\left(\theta(t)\right)(\omega) - \mathcal{F}(1)(\omega)\\ &= 2\left(\frac{1}{i\omega}+\pi\delta(\omega)\right) - 2\pi\delta(\omega) = \frac{2}{i\omega}\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}(\sin t)(\omega) &= \frac{1}{2i}\left(\mathcal{F}(e^{it})(\omega) - \mathcal{F}(e^{-it})(\omega)\right) = \frac{1}{2i}\left(2\pi\delta(\omega-1) - 2\pi\delta(\omega+1)\right)\\ &= \pi i(\delta(\omega+1) - \delta(\omega-1)).\end{aligned}$$

Using the convolution rule and the inversion formula we get

$$\mathcal{F}\left(\mathcal{F}(g) * \mathcal{F}(h)\right) = \mathcal{F}(\mathcal{F}(g)) \cdot \mathcal{F}(\mathcal{F}(h)) = 4\pi^2 g(-t)h(-t)$$

and

$$\mathcal{F}(g) * \mathcal{F}(h) = 2\pi\mathcal{F}(g \cdot h).$$

$$\begin{aligned}\mathcal{F}\left(\sin(|t|)\right)(\omega) &= \mathcal{F}\left(\sin t \cdot (2\theta(t)-1)\right)(\omega)\\ &= \frac{1}{2\pi}\big(\pi i\left(\delta(\omega+1) - \delta(\omega-1)\right)\big) * \frac{2}{i\omega}\\ &= \frac{1}{\omega+1} - \frac{1}{\omega-1}\\ &= \frac{2}{1-\omega^2}.\end{aligned}$$

Alternatively one can use Euler's formula for $\sin t = \frac{1}{2i}e^{it} - \frac{1}{2i}e^{-it}$ and use the rules for the Fouriertransform.