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MATEMATIK

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1.a) Setting $x_0 = 1, y_0 = 0$ and $x_0 = 0, y_0 = 1$ we get

$$e^{At} = \frac{1}{3}\begin{bmatrix} e^{5t} + 2e^{-t} & e^{5t} - e^{-t} \\ 2e^{5t} - 2e^{-t} & 2e^{5t} + e^{-t} \end{bmatrix}.$$

Since $A = \frac{d}{dt}e^{At}|_{t=0}$ we get

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

b) From the exponential matrix we can read of the eigenvalues of A:

$$\lambda_1 = 5, \qquad \lambda_2 = -1.$$

The corresponding eigenvectors are

$$s_1 = t_1(1,2)^T, t_1 \neq 0 \qquad \text{and} \qquad s_2 = t_2(1,-1)^T, t_2 \neq 0.$$

Then the general solution is given by

$$\begin{bmatrix} x_k \\ y_k \end{bmatrix} = c_1 5^k \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2(-1)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Expressing c_1, c_2 by means of the initial conditions we get

$$\begin{bmatrix} x_k \\ y_k \end{bmatrix} = \frac{1}{3}(a+b)5^k \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{3}(b-2a)(-1)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

2.a) We transform the equation with the one-sided Laplacetransform:

$$\begin{cases} sX(s) - 2 & = -3X(s) - Y(s) \\ sY(s) - 3 & = X(s) - Y(s) \end{cases}.$$

Solving this system we get

$$\begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} 2\frac{s+1}{(s+2)^2} - 3\frac{1}{(s+2)^2} \\ 2\frac{1}{(s+2)^2} + 3\frac{s+3}{(s+2)^2} \end{bmatrix}.$$

The inverse Laplacetransform gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} (2-5t)e^{-2t} \\ (3+5t)e^{-2t} \end{bmatrix}, \quad t > 0.$$

b) The systemmatrix has a double eigenvalue $\lambda = -2$. Therefore a generalized stationary solution exists for $s \neq -2$.

3.a) The productrule and the simplification rule $f(t)\delta_a(t) = f(a)\delta_a(t)$ gives

$$f'(t) = \cos t \cdot \theta(t+\pi/6) + \sin t \cdot \delta(t+\pi/6) = \cos t \cdot \theta(t+\pi/6) - \frac{1}{2}\delta(t+\pi/6).$$

b) A primitive funktion is $F(t) = \left(\frac{\sqrt{3}}{2} - \cos t\right)\theta(t+\pi/6) + C$. Villkoret $F(0) = 1$ ger $C = 2 - \frac{\sqrt{3}}{2}$. Therefore $F(t) = \left(\frac{\sqrt{3}}{2} - \cos t\right)\theta(t+\pi/6) + 2 - \frac{\sqrt{3}}{2}$.

c) Because $f'(t) - \cos t \cdot \theta(t+\pi/6) = -\dfrac{1}{2}\delta(t+\pi/6)$ and δ is a unit for convolution we get $(f' - \cos t \cdot \theta(t+\pi/6))) * f(t) = -\frac{1}{2}f(t+\pi/6) = -\frac{1}{2}\sin(t+\pi/6)\cdot\theta(t+\pi/3)$.

4.a) The system is linear, time-invariant and causal.

b) The system is **not** linear, time-invariant and causal.

c) The system is linear, **not** time-invariant and **not** causal.

5.a) The Fouriertransform of f is given by

$$\mathcal{F}(f)(\omega) = \int_{-2a}^{0} (2a+t)e^{-it\omega}\,dt + \int_{0}^{2a} (2a-t)e^{-it\omega}\,dt = 4\frac{\sin^2(a\omega)}{\omega^2}.$$

b) We will use Parseval's formula

$$16\int_{-\infty}^{\infty} \frac{\sin^4(a\omega)}{\omega^4}\,d\omega = 2\pi\int_{-2a}^{2a}(2a-|t|)^2\,dt = 4\pi\int_{0}^{2a}(2a-t)^2\,dt.$$

Hence,

$$\int_{-\infty}^{\infty}\frac{\sin^4(ax)}{x^4}\,dx = \frac{2\pi}{3}a^3.$$

6.a) We Fouriertransform the convolution equation

$$\frac{n}{2}\int_{-\infty}^{\infty} e^{-n|t-\tau|}y_n(\tau)\,d\tau = \frac{n}{2}\left(e^{-n|t|} * y_n(t)\right) = f(t)$$

and get

$$\frac{n}{2}\mathcal{F}\left(e^{-n|t|}\right)(\omega)\cdot\mathcal{F}(y_n(t))(\omega) = \mathcal{F}(f(t))(\omega).$$

We compute with the help of the scaling rule

$$\mathcal{F}\left(e^{-n|t|}\right)(\omega) = \frac{2n}{n^2+\omega^2}.$$

Therefore

$$\begin{aligned}
\mathcal{F}(y_n(t))(\omega) &= \frac{2}{n}\frac{\mathcal{F}(f)(\omega)}{\mathcal{F}\left(e^{-n|t|}\right)[\omega)} \\
&= \frac{\omega^2}{n^2}\mathcal{F}(f)(\omega) + \mathcal{F}(f)(\omega) \\
&= -\frac{1}{n^2}(i\omega)^2\mathcal{F}(f)(\omega) + \mathcal{F}(f)(\omega) \\
&= -\frac{1}{n^2}\mathcal{F}\left(\frac{d^2}{dt^2}f(t)\right)(\omega) + \mathcal{F}(f)(\omega) \\
&= \mathcal{F}\left(-\frac{1}{n^2}\frac{d^2}{dt^2}f(t) + f(t)\right)(\omega).
\end{aligned}$$

Hence,

$$y_n(t) = -\frac{1}{n^2} f(t) + f(t).$$

b) We immediately see from the solution

$$\lim_{n\to\infty} y_n(t) = \lim_{n\to\infty} \left(-\frac{1}{n^2} f(t) + f(t) \right) = f(t), \quad t \in \mathbb{R}.$$

Since f is a testfunction it is globally bounded, i.e. there is a number $M \in \mathbb{R}$ such that $M \geq |f(t)|, t \in \mathbb{R}$. Hence,

$$\|f(t) - y_n(t)\| \leq \frac{M}{n^2} \to 0,$$

and the convergence is uniform.