LUND UNIVERSITY

Centre for Mathematical Sciences
Mathematics, Faculty of Science

Ordinary Differential Equations II
Friday, March 17, 2017

Solutions

1. The fixed points satisfy the system

$$\begin{cases} 0 = x - z, \\ 0 = x(y+z) - 2z^2 + 1, \\ 0 = x^2 + y^3 - z^2. \end{cases}$$

The first equation gives $x = z$. Substituting this in the last equation gives $y = 0$ and we then find that $x^2 = 1$ from the second equation. Hence, the fixed points are $\pm(1, 0, 1)$.
The Jacobian of the right hand side is

$$\boldsymbol{f}'(x,y,z) = \begin{pmatrix} 1 & 0 & -1 \\ y+z & x & x-4z \\ 2x & 3y^2 & -2z \end{pmatrix}.$$

Hence

$$\boldsymbol{f}'(1,0,1) = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -3 \\ 2 & 0 & -2 \end{pmatrix}$$

with eigenvalues -1, 0 and 1, while

$$\boldsymbol{f}'(-1,0,-1) = \begin{pmatrix} 1 & 0 & -1 \\ -1 & -1 & 3 \\ -2 & 0 & 2 \end{pmatrix}$$

with eigenvalues -1, 0 and 3.
It follows that both fixed points are unstable.

2. Letting $L(x,y) = x^2 + y^2$ we find that the Lie derivative is

$$\dot{L}_{\boldsymbol{f}} = 2xy(x-1) + 2xy(1-x) - 2y^4 = -2y^4 \le 0.$$

Hence, L is a Liapunov function with respect to the origin since also $L(0,0) = 0$ and $L(x,y) > 0$ for $(x,y) \neq (0,0)$. It follows that the origin is a stable fixed point. To show that it's also asymptotically stable we use Krasovskii-LaSalle's theorem. Let $U = \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 < 1\}$ be the open unit disc. Note that $\dot{L}_{\boldsymbol{f}} = 0$ if and only if $y = 0$. When $y = 0$, we find that $y' = x(1-x) \neq 0$ unless $x = 0$ or 1. Hence, L isn't constant on any orbit lying in the punctured open unit disc $U \setminus \{(0,0)\}$. Moreover, the sublevel set $S_\delta = \{(x,y) \in U \colon L(x,y) \le \delta\} = \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 \le \delta\}$ is connected and compact for each $\delta \in (0,1)$. Hence, any solution starting in S_δ converges to $(0,0)$ as $t \to \infty$ and hence so does any solution starting in $U = \cup_{0<\delta<1} S_\delta$.

Please, turn over!

3. **a)** The eigenvalues of a Sturm-Liouville problem are real.
If $\lambda = -k^2 < 0$ we obtain that $y(x) = a\cosh(kx) + b\sinh(kx)$. The boundary condition at $x = 0$ gives $a = 0$. The boundary condition at $x = \pi/2$ then gives $bk\cosh(k\frac{\pi}{2}) = 0$ which yields $b = 0$. Hence, $y(x) \equiv 0$.
If $\lambda = 0$, we obtain $y(x) = ax + b$, and the boundary conditions give $a = b = 0$.
If $\lambda > k^2$, we obtain that $y(x) = a\cos(kx) + b\sin(kx)$. The boundary condition at $x = 0$ gives $y(x) = b\sin(kx)$. From the boundary condition at $x = \pi/2$ we obtain that $bk\cos(k\frac{\pi}{2}) = 0$, so $k = 2n + 1$, where n is an integer.
In conclusion, the eigenvalues are $\lambda_n = (2n+1)^2$ and the normalized eigenfunctions are $u_n(x) = \frac{2}{\sqrt{\pi}}\sin((2n+1)x)$, where $n = 0, 1, 2, \ldots$.

b) Let $\langle \cdot, \cdot \rangle$ denote the standard L^2 inner product on the interval $(0, \pi/2)$ and $\| \cdot \|$ the corresponding L^2 norm. We have

$$x = \sum_{n=0}^{\infty} c_n u_n(x),$$

where

$$\begin{aligned}
c_n &= \langle x, u_n \rangle \\
&= \frac{2}{\sqrt{\pi}} \int_0^{\frac{\pi}{2}} x\sin((2n+1)x)\,dx \\
&= \left[-\frac{2x\cos((2n+1)x)}{\sqrt{\pi}(2n+1)} \right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{2\cos((2n+1)x)}{\sqrt{\pi}(2n+1)}\,dx \\
&= \left[\frac{2\sin((2n+1)x)}{\sqrt{\pi}(2n+1)^2} \right]_0^{\frac{\pi}{2}} \\
&= \frac{2(-1)^n}{\sqrt{\pi}(2n+1)^2}.
\end{aligned}$$

Hence,

$$x = \sum_{n=0}^{\infty} \frac{4(-1)^n}{\pi(2n+1)^2}\sin((2n+1)x)$$

with convergence in L^2 norm. In fact, the series converges uniformly by Weierstrass' M-test

c) We have

$$\sum_{n=0}^{\infty} |c_n|^2 = \|x\|^2.$$

Hence,

$$\frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \int_0^{\frac{\pi}{2}} x^2\,dx = \frac{\pi^3}{24}$$

from which the result easily follows.

4. Let $r(t) = \sqrt{x^2(t) + y^2(t)}$, where $(x(t), y(t))$ is a solution. Then

$$\frac{d}{dt}r^2 = -2x^2 - 2y^2 + \frac{4(x^2+y^2)}{1+3x^2+2y^2} = \frac{2(x^2+y^2)}{1+3x^2+2y^2}(1 - 3x^2 - 2y^2),$$

so that $r(t)$ is increasing when $0 < 3x^2+2y^2 < 1$ and decreasing when $3x^2+2y^2 > 1$. In particular, the compact set $K = \{(x,y)\colon 1/3 \le x^2+y^2 \le 1/2\}$ is positively invariant (any annulus $c_1 \le x^2+y^2 \le c_2$ with $0 < c_1 \le 1/3$ and $c_2 \ge 1/2$ will work). The fixed points are the solutions of the system of equations obtained by setting $x' = y' = 0$. It is clear that the origin is a fixed point. Subtraction of x times the second equation from y times the first equation results in $x^2 + y^2 = 0$, showing that there are no other fixed points. The existence of a non-constant, closed orbit now follows from the Poincaré-Bendixson theorem.

5. Note that if x^z is defined as in the hint, then

$$\frac{d}{dx}x^z = \frac{z}{x}e^{z\log x} = ze^{(z-1)\log x} = zx^{z-1}.$$

It follows that x^z is a solution of $-u'' + \frac{c}{x^2}u = 0$, $x > 0$, if and only if

$$-z(z-1) + c = 0,$$

that is

$$z = \tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} + c}.$$

In particular, if $c < -\frac{1}{4}$, then $\sqrt{x}(\cos(b\log x) + i\sin(b\log x))$ is a solution of the equation if $b = \sqrt{|\frac{1}{4}+c|}$. Since the equation is real, so are $\sqrt{x}\cos(b\log x)$ and $\sqrt{x}\sin(b\log x)$. Both of these functions have zeros accumulating at $x = 0$ since $\log x \to -\infty$ as $x \to 0^+$.

If $q_0 < -1/4$ we can take $c \in (q_0, -1/4)$ and find that $q(x) < \frac{c}{x^2}$ for x sufficiently close to 0. But then we can use Sturm's comparison theorem, comparing with the solution $\sqrt{x}\cos(b\log x)$ of $-u'' + \frac{c}{x^2}u = 0$ with b defined as above, to find that a solution u of the equation $-u'' + q(x)u = 0$ has infinitely many zeros.

If $q_0 > 1/4$ we can take $c = -1/4$ and find that $q(x) > -\frac{1}{4x^2}$ for x sufficiently close to 0. We can now compare with the solution $\sqrt{x}$ of $-u'' - \frac{1}{4x^2}u = 0$, which has no zeros. Hence a non-trivial solution u of the equation $-u'' + q(x)u = 0$ can't have zeros accumulating at $x = 0$. It follows that it can have at most finitely many zeros in $(0,1]$. Otherwise the zeros would accumulate at some point $x_0 \in (0,1]$ and then we would have $u(x_0) = u'(x_0) = 0$ and hence $u(x) \equiv 0$ by Picard-Lindelöf's theorem.