LUND UNIVERSITY

Centre for Mathematical Sciences
Mathematics, Faculty of Science

Ordinary Differential Equations II
Wenesday, March 19, 2014
08.00-13.00

Solutions

1. The fixed points satisfy the system

$$\begin{cases} 2 - y - x^2 = 0 \\ 2x(x-y) = 0. \end{cases}$$

The second equation gives $x = 0$ or $x = y$. If $x = 0$, the first equation gives $y = 2$. If $x = y$, the first equation reduces to $x^2 + x - 2 = (x+2)(x-1) = 0$ with solutions $x = -2$ and $x = 1$. Hence, the fixed points are $(0,2)$, $(-2,-2)$ and $(1,1)$.
The Jacobian of the right hand side is

$$\boldsymbol{f}'(x,y) = \begin{pmatrix} -2x & -1 \\ 4x - 2y & -2x \end{pmatrix}.$$

Hence

$$\boldsymbol{f}'(0,2) = \begin{pmatrix} 0 & -1 \\ -4 & 0 \end{pmatrix}$$

with eigenvalues ± 2,

$$\boldsymbol{f}'(-2,-2) = \begin{pmatrix} 4 & -1 \\ -4 & 4 \end{pmatrix}$$

with eigenvalues 2 and 6, and

$$\boldsymbol{f}'(1,1) = \begin{pmatrix} -2 & -1 \\ 2 & -2 \end{pmatrix}$$

with eigenvalues $-2 \pm \sqrt{2}i$.
It follows that $(1,1)$ is asymptotically stable while $(0,2)$ and $(-2,-2)$ are unstable.

2. Consider the initial value problem

$$x' = 1 + \frac{x^2}{1+t^2}, \quad x(0) = 0.$$

a) We have

$$|f(t,x) - f(t,y)| = \left| \frac{x^2 - y^2}{1+t^2} \right| = \frac{|x+y|}{1+t^2}|x-y| \le 2|x-y|,$$

when $(t,x), (t,y) \in [0,1] \times [-1,1]$. Hence, f is Lipschitz continuous with respect to x with Lipschitz constant 2.

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b) We have that

$$M = \sup_{(t,x)\in[0,1]\times[-1,1]} |f(t,x)| = 2.$$

Hence, the IVP has a unique solution on the interval $[0, T_0]$, where

$$T_0 = \min\left\{T, \frac{\delta}{M}\right\} = \min\left\{1, \frac{1}{2}\right\} = \frac{1}{2},$$

by Picard-Lindelöf. In addition, the solution satisfies $|x(t)| \le \delta = 1$ there. To see that the solution is non-negative, we just note that $x' \ge 0$.

3. The corresponding system is

$$\begin{cases} x' = y, \\ y' = -x^3 - y^3. \end{cases}$$

Making the Ansatz $L(x, y) = ax^{2m} + by^{2n}$, we see that

$$L(x, y) = x^4 + 2y^2$$

is a Lyapunov function with

$$\frac{d}{dt}L(x, y) = -4y^4 \le 0$$

along a solution. Moreover, the derivative is zero only when $y = 0$. But then $y' = -x^3 \ne 0$ if $x \ne 0$. Hence the only orbit on which L is constant is the origin. It follows from Krasovskii-LaSalle's theorem that the origin is an asymptotically stable fixed point.

4. The homogeneous equation $y'' = 0$ has the general solution $y(x) = a + bx$. The homogeneous boundary condition $y(0) = 0$ gives $a = 0$, while the boundary condition $y'(1) = 0$ gives $b = 0$. Hence, the homogeneous problem only has the trivial solution $y(x) \equiv 0$. It follows that the problem in the exercise has a unique solution for each f. We can write the solution as $y(x) = y_1(x) + y_2(x)$, where y_1 solves the problem

$$\begin{aligned} &y_1''(x) = f(x), \quad 0 < x < 1, \\ &y_1(0) = 0, \\ &y_1'(1) = 0, \end{aligned}$$

and y_2 solves the problem

$$\begin{aligned} &y_2''(x) = 0, \quad 0 < x < 1, \\ &y_2(0) = 1, \\ &y_2'(1) = 2. \end{aligned}$$

To compute Green's function, we let $u_1(x) = x$ be a non-trivial solution of the equation which satisfies the homogeneous boundary condition at $x = 0$ and $u_2(x) = 1$ a non-trivial solution satisfying the homogeneous boundary condition at $x = 1$. The corresponding Wronskian is $W = u_1(x)u_2'(x) - u_1'(x)u_2(x) = -1$. Hence, Green's function is

$$G(x,\xi) = \begin{cases} \frac{u_1(\xi)u_2(x)}{a_2(\xi)W(\xi)}, & 0 \le \xi \le x \le 1 \\ \frac{u_1(x)u_2(\xi)}{a_2(\xi)W(\xi)}, & 0 \le x \le \xi \le 1 \end{cases}$$
$$= \begin{cases} -\xi, & 0 \le \xi \le x \le 1 \\ -x, & 0 \le x \le \xi \le 1, \end{cases}$$

and

$$y_1(x) = \int_0^1 G(x,\xi)f(\xi)\,d\xi = -\int_0^x \xi f(\xi)\,d\xi - \int_x^1 x f(\xi)\,d\xi.$$

On the other hand, $y_2(x) = a + bx$ for some a and b. Substituting this into the boundary conditions gives $a = 1$ and $b = 2$.

It follows that

$$y(x) = 1 + 2x + \int_0^1 G(x,\xi)f(\xi)\,d\xi = 1 + 2x - \int_0^x \xi f(\xi)\,d\xi - \int_x^1 x f(\xi)\,d\xi.$$

5. a) If $\lambda = -k^2 < 0$, we obtain that $y(x) = a\sinh(kx) + b\cosh(kx)$. The boundary condition at $x = 0$ gives $b = 0$ and without loss of generality we can take $y(x) = \sinh(kx)$. The boundary condition at $x = 1$ then reduces to $k = \tanh k$. Setting $f(k) = \tanh k - k$, we have that $f(0) = 0$ and $f'(k) = \tanh^2 k > 0$ for $k \ne 0$. Hence $f(k) \ne 0$ for $k \ne 0$. It follows that there are no negative eigenvalues.

If $\lambda = 0$, then $y(x) = a + bx$ for some a and b. The boundary conditions show that y is a solution iff $a = 0$. Hence, $\lambda_0 = 0$ is the first eigenvalue (with eigenfunction $u_0(x) = x$, say).

If $\lambda = k^2 > 0$, we obtain that $y(x) = a\sin(kx) + b\cos(kx)$ and from the boundary condition at $x = 0$ we get $b = 0$. We again take $a = 1$ so that $y(x) = \sin(kx)$. The boundary condition at $x = 1$ is then satisfied if and only if

$$\tan k = k.$$

It suffices to consider $k > 0$. The function $\tan k$ is strictly increasing from $-\infty$ to ∞ on each interval $\left((n - \frac{1}{2})\pi, \left(n + \frac{1}{2}\right)\pi\right)$, $n \in \mathbb{Z}$. Moreover, in this interval it is positive when $k \in \left(n\pi, \left(n + \frac{1}{2}\right)\pi\right)$. Hence, there is at least one zero in each interval $\left(n\pi, \left(n + \frac{1}{2}\right)\pi\right)$. To show that there is precisely one, we note that

$$\frac{d}{dk}(\tan k - k) = \tan^2 k > 0, \quad n\pi < k < \left(n + \tfrac{1}{2}\right)\pi.$$

It now follows that

$$\lambda_1 \in \left(\pi^2, \left(\tfrac{3\pi}{2}\right)^2\right), \lambda_2 \in \left((2\pi)^2, \left(\tfrac{5\pi}{2}\right)^2\right), \ldots, \lambda_n \in \left((n\pi)^2, \left(\left(n + \tfrac{1}{2}\right)\pi\right)^2\right), \ldots.$$

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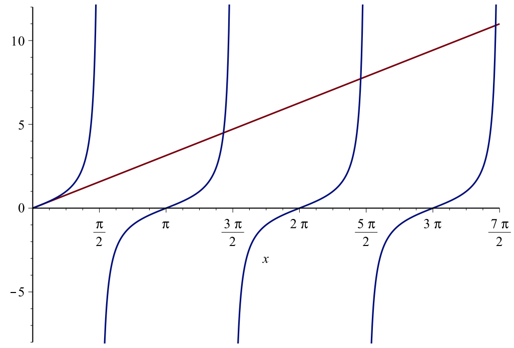

b) Let $k_n = \sqrt{\lambda_n}$ and $\delta_n := \left(n+\frac{1}{2}\right)\pi - k_n \in \left(0, \frac{\pi}{2}\right)$, $n \geq 1$. Note that $\tan k_n = k_n$ is equivalent to

$$\frac{1}{k_n} = \cot k_n.$$

We have

$$\cot k_n = \cot\left(\left(n+\tfrac{1}{2}\right)\pi - \delta_n\right) = \cot\left(\tfrac{\pi}{2} - \delta_n\right) = \tan\delta_n = \frac{1}{\left(n+\frac{1}{2}\right)\pi - \delta_n} \to 0$$

as $n \to \infty$. Hence $\delta_n \to 0$ as $n \to \infty$ (this is also easy to see geometrically). By Taylor expanding, we find that

$$\frac{1}{\left(n+\frac{1}{2}\right)\pi - \delta_n} = \tan\delta_n = \delta_n + \mathcal{O}(\delta_n^3).$$

Hence,

$$\delta_n(1+\mathcal{O}(\delta_n^2)) = \frac{1}{\left(n+\frac{1}{2}\right)\pi}\frac{1}{1+\mathcal{O}(\delta_n)}$$

and

$$\left(n+\tfrac{1}{2}\right)\pi\,\delta_n = \frac{1}{1+\mathcal{O}(\delta_n)} \to 1$$

as $n \to \infty$. But this means that

$$\begin{aligned}
\lambda_n - \left(\left(n+\tfrac{1}{2}\right)\pi\right)^2 &= k_n^2 - \left(\left(n+\tfrac{1}{2}\right)\pi\right)^2 \\
&= \left(\left(n+\tfrac{1}{2}\right)\pi - \delta_n\right)^2 - \left(\left(n+\tfrac{1}{2}\right)\pi\right)^2 \\
&= \left(\left(n+\tfrac{1}{2}\right)\pi\right)^2 - 2\left(n+\tfrac{1}{2}\right)\pi\,\delta_n + \delta_n^2 - \left(\left(n+\tfrac{1}{2}\right)\pi\right)^2 \\
&= -2\left(n+\tfrac{1}{2}\right)\pi\,\delta_n + \delta_n^2 \\
&\to -2
\end{aligned}$$

as $n \to \infty$.