LUND UNIVERSITY

Written Examination
Ordinary Differential Equations II
Friday, March 17, 2017
08.00-13.00

Centre for Mathematical Sciences
Mathematics, Faculty of Science

Note: Only students who are registered or re-registered on the course are allowed to take the exam.

No aids allowed. Use the distributed paper sheets and write only on one side. Fill in the cover sheet completely. Write legibly (in Swedish or English). Motivate your conclusions clearly and concisely; draw a picture if appropriate.

Test results: *Posted Monday, March 20, before 17.00. Viewing of marked exam scripts: Tuesday, March 21, 11.30-12.00 in room 508.*

Oral exams: *Wednesday, March 22 – Monday, March 27. State your preference (day and AM/PM) on the cover sheet of your test – at least two options.*

1. Find all fixed points of the system

$$\begin{cases} x' = x - z, \\ y' = x(y+z) - 2z^2 + 1, \\ z' = x^2 + y^3 - z^2. \end{cases}$$

Determine whether they are stable, asymptotically stable or unstable.

2. The system

$$\begin{cases} x' = y(x-1), \\ y' = x(1-x) - y^3 \end{cases}$$

has a fixed point at the origin. Show that it is asymptotically stable and that every solution starting in the open unit disc $\{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 < 1\}$ converges to the origin as $t \to \infty$.

3. **a)** Find the eigenvalues and normalized eigenfunctions of the Sturm-Liouville problem

$$\begin{aligned} -y'' &= \lambda y, \qquad 0 < x < \tfrac{\pi}{2}, \\ y(0) &= y'(\tfrac{\pi}{2}) = 0. \end{aligned}$$

b) Expand the function $f(x) = x$ using these eigenfunctions. In which sense does the series converge?

c) Show that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.$$

Please, turn over!

4. Show that the system

$$\begin{cases} x' = -x + y + \frac{2x}{1+3x^2+2y^2}, \\ y' = -x - y + \frac{2y}{1+3x^2+2y^2} \end{cases}$$

has a regular periodic orbit.

5. Consider the equation

$$-u'' + q(x)u = 0, \quad 0 < x \leq 1,$$

where $q \in C(0,1]$ with $x^2 q(x) \to q_0$ as $x \to 0^+$. Show that any non-trivial solution has infinitely many zeros if $q_0 < -1/4$ and finitely many zeros if $q_0 > -1/4$.

Hint: Show that for any $c \in \mathbb{R}$, the equation $-u'' + \frac{c}{x^2}u = 0$, $x > 0$, has a solution of the form $u(x) = x^z$ for some constant $z \in \mathbb{C}$ (which depends on q_0). Here x^z is defined as $e^{z \log x}$ for $x > 0$, where $\log x$ is the natural logarithm of x. In particular, $x^{a+ib} = x^a(\cos(b \log x) + i \sin(b \log x))$ if $a, b \in \mathbb{R}$.