LUND UNIVERSITY

Written Examination
Ordinary Differential Equations II
Wenesday, March 19, 2014
08.00-13.00

Centre for Mathematical Sciences
Mathematics, Faculty of Science

Note: Only students who are registered or re-registered on the course are allowed to take the exam.

No aids allowed. Use the distributed paper sheets and write only on one side. Fill in the cover sheet completely and write your initials on each sheet. Write legibly (in Swedish or English). Motivate your conclusions clearly and concisely; draw a picture if appropriate.

Test results: *Posted Friday, March 21, before 17.00.*

Oral exams: *Monday, March 24 – Friday, March 28. State your preference (day and AM/PM) on the cover sheet of your test – at least two options.*

1. Find all fixed points of the system

$$\begin{cases} x' = 2 - y - x^2 \\ y' = 2x(x-y). \end{cases}$$

Determine whether they are stable, asymptotically stable or unstable.

2. Consider the initial value problem

$$x' = 1 + \frac{x^2}{1+t^2}, \quad x(0) = 0.$$

a) Show that the function $f(t,x) = 1 + \frac{x^2}{1+t^2}$ is Lipschitz continuous with respect to x on the set $[0,1] \times [-1,1]$ and determine a Lipschitz constant.

b) Show that there is a unique solution on the interval $\left[0, \frac{1}{2}\right]$, which satisfies $0 \leq x(t) \leq 1$.

3. Prove that the origin is an asymptotically stable fixed point for the system corresponding to the equation

$$x'' + (x')^3 + x^3 = 0.$$

4. Show that the boundary value problem

$$\begin{aligned} &y''(x) = f(x), \quad 0 < x < 1, \\ &y(0) = 1, \\ &y'(1) = 2 \end{aligned}$$

has a unique solution for each $f \in C[0,1]$. Express the solution using Green's function.

Please, turn over!

5. Consider the Sturm-Liouville eigenvalue problem

$$\begin{aligned} -y'' &= \lambda y, \\ y(0) &= 0, \\ y'(1) - y(1) &= 0. \end{aligned}$$

Let $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ be the eigenvalues, ordered increasingly.

a) Show that $\lambda_0 = 0$ and that $(n\pi)^2 < \lambda_n < \left(\left(n + \frac{1}{2}\right)\pi\right)^2$, $n \geq 1$.

b) Show that

$$\lim_{n\to\infty} \left(\lambda_n - \left(\left(n + \tfrac{1}{2}\right)\pi\right)^2\right) = -2.$$