LUNDS TEKNISKA HÖGSKOLA
MATEMATIK

TENTAMENSSKRIVNING
DISKRET MATEMATIK
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SOLUTIONS

1. **a)** This equals the number of ways we can arrange 12 circles and 3 dividers and hence equals $\binom{15}{3} = 455$.

b) By letting $y_1 = x_1 - 5$ and $y_i = x_i + 1$ for $2 \le i \le 4$ we obtain the equivalent problem of finding the number of non-negative solutions to $y_1+y_2+y_3+y_4 = 10$. As above we find that this equals $\binom{13}{3} = 286$.

c) Using generating functions we try to find the coefficient of x^{12} in

$$f(x) = (1+x^2+x^4+x^6+\cdots)(1+x+x^2+x^3+\cdots)^3 = (1+x^2+x^4+x^6+\cdots)\sum_{j=0}^{\infty}\binom{-3}{j}(-x)^j$$

Hence the sought after coefficent equals $\sum_{k=0}^{6}\binom{-3}{2k}(-1)^{2k} = \sum_{k=0}^{6}\binom{2k+2}{2} = \binom{2}{2}+\binom{4}{2}+\binom{6}{2}+\binom{8}{2}+\binom{10}{2}+\binom{12}{2}+\binom{14}{2} = \frac{2\cdot1+4\cdot3+6\cdot5+8\cdot7+10\cdot9+12\cdot11+14\cdot13}{2} = 252$

2. Use the principle of inclusion/exclusion with the ground space consisting of all arrangements of the letters in COOKBOOKS, and the conditions $c_1 =$ the word contains COOK, $c_2 =$ the word contains *BOOK* and $c_3 =$ the word contains SOCK. Then $N(\bar{c_1}\bar{c_2}\bar{c_3}) = S_0 - S_1 + S_2 - S_3$ where $S_0 = \frac{9!}{4!2!} = 7560$, the size of the ground set. Further $S_1 = N(c_1)+N(c_2)+N(c_3) = \frac{6!}{2!}+\frac{6!}{2!}+\frac{6!}{3!} = 360+360+120 = 840$. (For example, to obtain $N(c_2)$ count the arrangments of BOOK, C, O, O, K, S regarding BOOK as one unit.) Next we find that $S_2 = N(c_1c_2) + N(c_1c_3) + N(c_2c_3) = 3! + 0 + 3! = 12$ since conditions one and three are incompatible. Finally $S_3 = N(c_1c_2c_3) = 0$ and we get our result

$$N(\bar{c_1}\bar{c_2}\bar{c_3}) = S_0 - S_1 + S_2 - S_3 = 7560 - 840 + 12 - 0 = 6732.$$

3. **a)** $s_{n+1} - s_n = \binom{n+1}{2} = \frac{n^2}{2} + \frac{n}{2}$ with $s_2 = 1$

b) The homogenous equations is solved by any constant so we have $s_n^{(h)} = A$. To find a particular solution we substitute $s_n^{(p)} = Bn^3 + Cn^2 + Dn$ into the equation and find by equating coefficients that $B = 1/6$, $C = 0$, and $D = -1/6$. Then all solutions are given by $s_n = s_n^{(h)} + s_n^{(p)} = A + \frac{n^3}{6} - \frac{n}{6}$. Using our initial condition we find that $A = 0$ and hence that $s_n = \frac{n^3}{6} - \frac{n}{6}$.

4. We first note that 5, 6 and 7 are relatively prime and therefore the Chinese remainder theorem applies if we can transform the equations to the form $x \equiv a_i$ modulo n_i. For example in $\mathbb{Z}_5$ we have that $3x + 2 = 1 \Leftrightarrow 3x = 4 \Leftrightarrow x = 3$ where the second step comes from multiplying with $3^{-1} = 2$. Doing this for the other two equations we get the equivalent system

$$\begin{cases} x \equiv 3 \pmod 5 \\ x \equiv 3 \pmod 6 \\ x \equiv 2 \pmod 7. \end{cases}$$

Using the method from the proof of the Chinese remainder theorem we get a solution $x = a_1s_1N_1 + a_2s_2N_2 + a_3s_3N_3 = 3\cdot(-2)\cdot 42 + 3\cdot(-1)\cdot 35 + 2\cdot(-3)\cdot 30 = -252 - 105 - 180 \equiv 93 \pmod{210}$. By the Chinese remainder theorem we know that all solutions are given by $x = 93 + 210n$ where n is any integer.

5. **a)** A polynomial $h(x) = x^3 + ax^2 + bx + c$ in $\mathbb{Z}_3[x]$ is irreducible if and only if it has no zeroes, that is if

$$\begin{cases} h(0) = c \neq 0 \\ h(1) = 1 + a + b + c \neq 0 \\ h(-1) = -1 + a - b + c \neq 0. \end{cases}$$

It is easy to find solutions (a, b, c) and hence get irreducible polynomials. Two exapmles are $(a, b, c) = (1, 0, 2)$ and $(a, b, c) = (2, 0, 1)$ resulting in the polynomials $h_1(x) = x^3 + x^2 + 2$ and $h_2(x) = x^3 + 2x^2 + 1$. To see how many solutions there are look at the system:

$$\begin{cases} h(0) = c = d \\ h(1) = 1 + a + b + c = e \\ h(-1) = -1 + a - b + c = f. \end{cases}$$

Here there are $2^3 = 8$ choices of non-zero parameters (d, e, f) and for given (d, e, f) the system has a unique solution. (We can see this for example by computing the determinant.) Hence there are 8 irreducible polynomials of degree three.

b) Let us use $h(x) = h_1(x) = x^3 + x^2 + 2$. Then the elements of the field are all equivalency classes $[ax^2+bx+c]$ where $a, b, c \in \mathbb{Z}_3$ addition and multiplcation is done by adding or multiplying representatives of the classes and then finding a representative of degree at most two by taking the remainder after division by h. In the given examples $[x^2+2x]+[x+1] = [x^2+3x+1] = [x^2+1]$, $[x^2 + x] \cdot [x^2 + 2] = [x^4 + x^3 + 2x^2 + 2x] = [x(x^3 + x^2 + 2) + 2x^2] = [2x^2]$. Inverses can be found using the Euclidean algorithm. By division $x^3 + x^2 + 2 = x^2(x + 1) + 2$. Rearranging and multiplying by 2 we find that $1 = 2(x^3 + x^2 + 2) + x^2(-2x - 2)$ which shows that $[-2x - 2] = [x + 1]$ is an inverse of $[x^2]$.

6. **a)** The code is an 8-dimensional subspace and therefore contains 11^8 words.

b) Looking at the columns of H we can see that any two of them are independent but for example the first three columns are linearly dependent. This shows that the separation is three.

c) The syndromes of the words are $(0\,0)$, $(4\,0)$ and $(4\,9)$ respectively. This shows that w_1 is a code word and the other two are not. Moreover w_2 is not a multiple of a column in H and therefore it must contain at least two errors. Since there are several ways to write $(4\,0)$ as a linear combination of two columns we cannot properly correct w_2. Assume $(4\ 9)$ is a multiple of column k. Then $(4\ 9) = a(1\ k)$ and hence $a = 4$ and $k = 4^{-1} \cdot 9 = 3 \cdot 9 = 27 = 5$ in $\mathbb{Z}_{11}$. This shows that w_3 can be corrected by subtracting 4 from the element in position 5. This results in the corrected word $w_3 = (1\,0\,0\,1\,0\,0\,1\,0\,4\,4)$