Numerical Methods for Differential Equations exam 2017-04-18
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Exam duration 14:00 – 18:00. Place: MA8:B. A minimum of 16 points out of 32 are required to pass. Your grade is determined by the sum of your exam and project scores, in accordance with the rules published on the course home page.

No computers, pocket calculators, cell phones, browsing tablets or any other electronic devices, and no textbooks, lecture notes or written material, may be used during the exam. Only paper, pen or pencil, and personal competence, are permitted.

1. **(5p)** Consider the multistep formula

$$y_{n+1} - \frac{\alpha}{2}y_n - \frac{1}{2}y_{n-1} = h\big[\beta_2 f(y_{n+1}) + \beta_1 f(y_n) + \beta_0 f(y_{n+1})\big]$$

for the initial value problem

$$y' = f(y); \qquad y(0) = y_0.$$

   (a) Determine the value of the parameter α so that the method is *consistent.* (1p) $\alpha = 1$

   (b) Is the method *zero-stable*? (Motivate your answer.) (1p) Yes $\rho = (w-1)(w+\frac{1}{2})$

   (c) Determine the coefficients β_i so that *the order of the method is maximal*, and give the maximal order. (3p) $p = 3$ $\beta_0 = \frac{1}{8}$ $\beta_1 = 1$ $\beta_2 = \frac{3}{8}$

2. **(5p)** Consider the 3-stage Runge-Kutta method with Butcher tableau

0	0	0	0
1/2	1/2	0	0
3/4	0	3/4	0
	2/9	3/9	4/9

   (a) Write the formulas associated with this method and find its stability function $R(h\lambda)$. (3p) $R(h\lambda) = 1 + h\lambda + \frac{(h\lambda)^2}{2} + \frac{(h\lambda)^3}{6}$

   (b) Calculate $|R(ih\omega)|$ and show whether the method is A-stable or not. Give a simple argument for why this is the case. (2p) Not A-stable. R is a polynomial

3. **(4p)** Construct a second order discretization of the nonlinear two-point boundary value problem $\Delta x = 1/(N+1)$

$$u'' - uu' + u = f(x)$$
$$u(0) = 0, \qquad u(1) = 0.$$

$$\frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} - u_j \frac{u_{j+1} - u_{j-1}}{2\Delta x} + u_j = f(x_j)$$

$$J = \left[\frac{1}{\Delta x^2} + \frac{u_j}{2\Delta x} \; ; \; -\frac{2}{\Delta x^2} - \frac{u_{j+1} - u_{j-1}}{2\Delta x} + 1 \; ; \; \frac{1}{\Delta x^2} - \frac{u_j}{2\Delta x} \right]$$

(a) Introduce a grid and discretize with a standard second order method. Be careful to explain your notation, and how Δx is related to the number of equations, N. (2p)

(b) As the system is nonlinear we need to solve it using Newton's method. Find the Jacobian matrix of the system. (2p)

4. **(4p)** In order to determine the shape of a circular drum skin's radial oscillation modes, we need to solve the *Bessel equation*

$$-\frac{(xu')'}{x} = \lambda u$$

with boundary conditions $u'(0) = 0$, $u(R) = 0$, where R is the radius of the drum membrane.

Construct a second order discretization, representing the boundary conditions to 2nd order accuracy. Write down details of where the grid points are located, and how you select the mesh width Δx so that the problem is ready for programming. State the linear algebraic eigenvalue problem in matrix-vector form.

5. **(5p)** Let $u(x)$ be a differentiable function on $[0, 1]$ with periodic boundary conditions $u(0) = u(1)$, and let the inner product $\langle \cdot, \cdot \rangle$ be defined by

$$\langle u, v \rangle = \int_0^1 uv \, \mathrm{d}x.$$

$\langle u', u^p \rangle = [u^{p+1}]_0^1 - \langle u, p u^{p-1} u' \rangle$
$= -p \langle u^p, u' \rangle = 0$

(a) Show that u' is orthogonal to u^p, for $p \geq 1$. (2p)

(b) Consider the inviscid Burgers equation $u_t - uu_x = 0$ with periodic boundary conditions and assume that the solution is continuously differentiable. Show that $\|u(t, \cdot)\|_2$ remains constant for all t. (3p)

$\langle u, u_t \rangle = \langle u^2, u_x \rangle = 0 \Rightarrow \|u(t,\cdot)\|_2 = \text{const.}$

6. **(5p)** For $t \geq 0$ and $x \in [0, 1]$, let u_j^n approximate $u(j \cdot \Delta x, n \cdot \Delta t)$. Write down the differential equations corresponding to the discretizations below, and give the name of the equation in each case.

(a)

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2} + \frac{u_{j+1}^n - u_j^n}{\Delta x}$$

$u_t = u_{xx} + u_x$
conv-diff.

(b)

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{u_j^n - u_{j-1}^n}{\Delta x} = 0$$

$u_t + u_x = 0$
advection

(c)

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + u_j^n \cdot \frac{u_{j+1}^n - u_j^n}{\Delta x} = 0$$

$u_t + uu_x = 0$
inviscid Burgers

(d)

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

$u_{tt} = u_{xx}$ wave

(e)

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2} + f(u_j^n)$$

$u_t = u_{xx} + f(u)$
reaction-diff.

7. **(4p)** The *central difference scheme* for the advection equation with $a > 0$ and periodic boundary conditions,

$$\begin{aligned} &u_t + au_x = 0, \quad 0 \le x \le 1, \quad t \ge 0, \\ &u(x, 0) = \phi_0(x), \end{aligned}$$

is

$$u_j^{n+1} = u_j^n - a\Delta t(u_{j+1}^n - u_{j-1}^n)/(2\Delta x).$$

(a) Let $\mu = \Delta t/\Delta x$. Rewrite this scheme as a matrix–vector recursion $U^{n+1} = T_\mu U^n + V^n$ with $U^n = [u_1^n, u_2^n, \dots, u_L^n]^{\mathrm{T}}$, giving the matrix T_μ and the vector V^n. Is the matrix symmetric, skew-symmetric, unsymmetric, circulant, or of some other type? (2p)

circulant

(b) The central difference method is best known for always being unstable. Propose a change of the scheme so that you obtain a stable, explicit method. (2p)

upwind

L-W

L-F methods all fine

LYCKA TILL — GOOD LUCK! G.S.

$$T = \begin{bmatrix} 1 & a\mu/2 & & & -\frac{a\mu}{2} \\ -a\mu/2 & 1 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{a\mu}{2} \\ \frac{a\mu}{2} & & & -\frac{a\mu}{2} & 1 \end{bmatrix}$$

$$hY_1' = h\lambda$$

$$hY_2' = h\lambda\left(1+\frac{h\lambda}{2}\right) = h\lambda + \frac{(h\lambda)^2}{2}$$

$$hY_3' = h\lambda\left(1+\frac{3h\lambda}{4} + \frac{3(h\lambda)^2}{8}\right) = h\lambda + \frac{3}{4}(h\lambda)^2 + \frac{3}{8}(h\lambda)^3$$

$$y_1 = 1 + \frac{2}{9}h\lambda + \frac{3}{9}\left(h\lambda + \frac{(h\lambda)^2}{2}\right) + \frac{4}{9}\left(h\lambda + \frac{3}{4}(h\lambda)^2 + \frac{3}{8}(h\lambda)^3\right)$$

$$= 1 + h\lambda\underbrace{\left(\frac{2}{9}+\frac{3}{9}+\frac{4}{9}\right)}_{1} + \frac{(h\lambda)^2}{2}\underbrace{\left(\frac{3}{9}+\frac{12}{18}\right)}_{1} + \frac{(h\lambda)^3}{6}\underbrace{\left(\frac{\overbrace{3\cdot4\cdot6}^{3\cdot3\cdot4\cdot2}}{9\cdot8}\right)}_{1}$$

$$= 1 + h\lambda + \frac{(h\lambda)^2}{2} + \frac{(h\lambda)^3}{6}$$