

**Numerical Methods for Differential Equations FMNN10/NUMN12**  
**Exam 2015-01-12** **Answers**

Exam duration 14:00 – 19:00. A minimum of 16 points out of 32 are required to pass. Your grade is determined by the sum of your exam and project scores, in accordance with the rules on the course home page.

No computers, pocket calculators, cell phones, browsing tablets or any other electronic devices, and no textbooks, lecture notes or written material, may be used during the exam.

1. (5p) You are familiar with the standard BDF methods, e.g.

$$\frac{3}{2}y_{n+1} - 2y_n + \frac{1}{2}y_{n-1} = hf(y_{n+1})$$

for the initial value problem

$$y' = f(y); \quad y(0) = y_0.$$

The BDF methods are implicit and approximate  $y'_{n+1}$  by a linear combination of  $y$  values. One can also construct *explicit* methods based on backward differences that approximate  $y'_{n+1}$ . To the two-step BDF above there corresponds an explicit two-step method

$$\frac{3}{2}y_{n+1} - 2y_n + \frac{1}{2}y_{n-1} = h\beta_1 f(y_n) + h\beta_0 f(y_{n-1}).$$

- (a) Show that this method is *zero-stable*. (1p)
- (b) Determine the values of the parameters  $\beta_0$  and  $\beta_1$  so that the method is of maximal order. What order can it reach? (3p)
- (c) Is the resulting method A-stable? (Motivate your answer.) (1p)

**Ans.**

- (a) For zero stability, the  $\rho$  polynomial must satisfy the root condition. The characteristic equation is

$$\frac{3}{2}w^2 - 2w + \frac{1}{2} = 0$$

with roots  $w = 1$  and  $w = 1/3$ , so the method is zero-stable.

- (b) An explicit zero-stable two-step method can at most reach order  $p = 2$ . To make it a 2nd order method, we construct the order conditions that determine  $\beta_0$  and  $\beta_1$ . Insert polynomials 1,  $t$ , and  $t^2$  (with derivatives 0, 1 and  $2t$ , respectively), to get

$$\begin{aligned}\frac{3}{2} - 2 + \frac{1}{2} &= h\beta_1 \cdot 0 + h\beta_0 \cdot 0 \\ \frac{3}{2} \cdot (2h) - 2 \cdot (h) + \frac{1}{2} \cdot 0 &= h\beta_1 \cdot 1 + h\beta_0 \cdot 1 \\ \frac{3}{2} \cdot (2h)^2 - 2 \cdot (h)^2 + \frac{1}{2} \cdot 0 &= h\beta_1 \cdot (2h) + h\beta_0 \cdot 0\end{aligned}$$

The first equation is obviously satisfied, and the last two give

$$\begin{aligned}3h - 2h &= h(\beta_1 + \beta_0) \\ 6h^2 - 2h^2 &= h^2(2\beta_1)\end{aligned}$$

from which we conclude  $\beta_1 = 2$ ,  $\beta_0 = -1$ .

- (c) As the method is explicit it cannot be A-stable.

2. (5p) Consider the 3-stage Bogacki-Shampine Runge-Kutta method

$$\begin{aligned}Y_1' &= f(t_n, y_n) \\ Y_2' &= f(t_n + h/2, y_n + hY_1'/2) \\ Y_3' &= f(t_n + 3h/4, y_n + 3hY_2'/4) \\ y_{n+1} &= y_n + h(2Y_1' + 3Y_2' + 4Y_3')/9\end{aligned}$$

- (a) Construct its Butcher tableau. (1p)  
 (b) Apply the method to the linear test equation  $y' = \lambda y$  and construct the method's *stability function*  $P(h\lambda)$  and simplify it as far as possible. (2p)  
 (c) Without making further investigations, what is the method's apparent order, and is the method A-stable? (Motivate your answer.) (2p)

**Ans.** The Butcher tableau is

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 3/4 & 0 & 3/4 & 0 \\ \hline & 2/9 & 3/9 & 4/9 \end{array}$$

Applying the method to  $y' = \lambda y$  gives

$$\begin{aligned} hY'_1 &= h\lambda \\ hY'_2 &= h\lambda(1 + h\lambda/2) \\ hY'_3 &= h\lambda(1 + 3h\lambda/4 + 3(h\lambda)^2/8) \end{aligned}$$

so

$$\begin{aligned} P(h\lambda) &= 1 + \frac{1}{9}(2h\lambda + 3h\lambda + 3(h\lambda)^2/2 + 4h\lambda + 3(h\lambda)^2 + 3(h\lambda)^3/2) \\ &= 1 + h\lambda + (h\lambda)^2/2 + (h\lambda)^3/6. \end{aligned}$$

The method's order is apparently 3, as the stability polynomial recovers the Taylor expansion of the exponential function up to the third order term. As the stability function is a polynomial, the method cannot be A-stable, as  $P(\infty) = \infty$ .

3. **(4p)** Consider the stationary convection–diffusion equation

$$\begin{aligned} u'' + \alpha u' &= g(x) \\ u(0) &= 1, \quad u(1) = 1. \end{aligned}$$

- (a) Introduce a suitable grid and discretize with a *standard second order method*. Give *all details about the grid*, such as the number of grid points and their location, as well as mesh width  $\Delta x$ , and formulate the discretization. Include the boundary conditions in the equation system. (2p)
- (b) Construct the matrix associated with the discretization. Is it a Toeplitz matrix? Is it symmetric, skew-symmetric, circulant, or of some other structure? (2p)

**Ans.** With  $N$  interior points  $x_j = j\Delta x$  and  $\Delta x = 1/(N + 1)$ , the discretization is

$$\begin{aligned} \frac{-2u_1 + u_2}{\Delta x^2} + \alpha \frac{u_2}{2\Delta x} &= g(x_1) - \frac{1}{\Delta x^2} + \frac{\alpha}{2\Delta x} \\ \frac{u_{j-1} - 2u_j + u_{j+1}}{\Delta x^2} + \alpha \frac{-u_{j-1} + u_{j+1}}{2\Delta x} &= g(x_j) \\ \frac{u_{N-1} - 2u_N}{\Delta x^2} + \alpha \frac{-u_{N-1}}{2\Delta x} &= g(x_N) - \frac{1}{\Delta x^2} - \frac{\alpha}{2\Delta x}. \end{aligned}$$

The matrix is tridiagonal Toeplitz (neither symmetric, skew-symmetric nor circulant) and becomes

$$= \frac{1}{\Delta x^2} \begin{pmatrix} -2 & 1 + \alpha\Delta x/2 & & & \\ 1 - \alpha\Delta x/2 & -2 & & 1 + \alpha\Delta x/2 & \\ & & \ddots & & \\ & & & & 1 - \alpha\Delta x/2 & -2 \end{pmatrix}$$

4. (5p) Consider the eigenvalue problem associated with the stationary convection–diffusion operator above, i.e.,

$$\mathcal{L}_\alpha u = \lambda u,$$

with homogeneous boundary conditions  $u(0) = u(1) = 0$  on  $[0, 1]$ , and where  $\mathcal{L}_\alpha u = u'' + \alpha u'$ .

- (a) Solve the *analytical eigenvalue* problem and find its eigenvalues  $\lambda_k$  and corresponding eigenfunctions  $u_k(x)$ . (3p)  
 (b) Construct an approximating algebraic eigenvalue problem

$$Au = \lambda_{\Delta x} u.$$

by using a *second order discretization* using symmetric difference quotients to approximate the derivatives, and give the matrix  $A$ . Given that the eigenvalues of an non-symmetric tridiagonal  $N \times N$  Toeplitz matrix  $A = \text{tridiag}(b \ a \ c)$  are

$$\lambda_k = a + 2\sqrt{bc} \cos \frac{k\pi}{N+1}; \quad k = 1 : N,$$

and that the analytic eigenvalues of the differential operator  $\mathcal{L}_\alpha$  are *real*, what is the largest value of  $\Delta x$  (in terms of other parameters) you can choose so that the discrete problem has the same property? (2p)

**Ans.**

- (a) The analytic eigenvalue problem  $u'' + \alpha u' = \lambda u$  has characteristic equation  $\kappa^2 + \alpha\kappa - \lambda = 0$  with roots

$$\kappa = -\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} + \lambda}.$$

Hence  $u(x) = e^{-\alpha x/2} (Ae^{\gamma x} + Be^{-\gamma x})$ , where  $\gamma$  represents the square root. Imposing the boundary condition  $u(0) = 0$  shows that  $A = -B$ , so the eigenfunctions are

$$u(x) = e^{-\alpha x/2} (e^{\gamma x} - e^{-\gamma x}),$$

which cannot satisfy the condition  $u(1) = 0$  unless  $\gamma$  is imaginary. So we write  $\gamma = i\omega$  to get the eigenfunctions

$$u_k(x) = e^{-\alpha x/2} \sin \omega_k x,$$

which satisfies  $u(1) = 0$  iff  $\omega_k = k\pi$ , implying that

$$u_k(x) = e^{-\alpha x/2} \sin k\pi x.$$

The eigenvalues are determined from the square root by

$$-\omega_k^2 = -k^2\pi^2 = \lambda_k + \frac{\alpha^2}{4},$$

so

$$\lambda_k = -(k\pi)^2 - \frac{\alpha^2}{4}.$$

(b) The matrix is the same as in the previous problem, i.e.,

$$A = \frac{1}{\Delta x^2} \begin{pmatrix} -2 & 1 + \alpha\Delta x/2 & & & \\ 1 - \alpha\Delta x/2 & -2 & 1 + \alpha\Delta x/2 & & \\ & & \ddots & & \\ & & & 1 - \alpha\Delta x/2 & -2 \end{pmatrix}$$

Its eigenvalues are, according to the given formula

$$\lambda_k[A] = -\frac{2}{\Delta x^2} + \frac{2}{\Delta x^2} \sqrt{1 - \frac{\alpha^2 \Delta x^2}{4}} \cos \frac{k\pi}{N+1}$$

The eigenvalues are real iff  $|\alpha\Delta x| < 2$  (this is also known as the mesh Péclet number), which means that one must choose  $\Delta x \leq 2/|\alpha|$ , or, equivalently,  $N+1 \geq |\alpha|/2$ .

5. **(5p)** Classify the following PDEs for  $t \geq 0$  and  $x \in [0, 1]$ :

- (a)  $u_t + uu_x = 0$
- (b)  $u_t + a \cdot u_x = 0$
- (c)  $u_t = (d(u)u_x)_x$
- (d)  $u_{tt} = c^2 \cdot u_{xx}$
- (e)  $u_t = d \cdot u_{xx} + f(u)$

For each equation, state whether the problem is *elliptic*, *parabolic* or *hyperbolic*. All parameters are supposed to be *positive*. In addition, give the “name” of each equation (e.g. “convection–diffusion equation” etc.).

**Ans.** Problem a) is the hyperbolic inviscid Burgers equation; b) is the hyperbolic advection equation; c) is a parabolic nonlinear diffusion equation; d) is the hyperbolic wave equation; e) is the parabolic reaction–diffusion equation.

6. **(4p)** Once again, consider the stationary convection–diffusion operator  $\mathcal{L}_\alpha u = u'' + \alpha u'$  with homogeneous Dirichlet boundary conditions  $u(0) = u(1) = 0$ , and let the inner product  $\langle \cdot, \cdot \rangle$  be defined by

$$\langle u, v \rangle = \int_0^1 uv \, dx.$$

The operator  $\mathcal{L}_\alpha$  is not self-adjoint with the respect to the given inner product. However, one can find a function  $w(x)$  such that the operator  $w\mathcal{L}_\alpha$  is self-adjoint.

- (a) Determine  $w$  such that  $(wu')' = w(u'' + \alpha u')$ . (2p)  
 (b) Use integration by parts to determine the logarithmic norm of  $\mathcal{L}_\alpha$ , i.e., find the smallest value of the real-valued constant  $\mu_2[\mathcal{L}_\alpha]$  such that  $\langle u, \mathcal{L}_\alpha u \rangle \leq \mu_2[\mathcal{L}_\alpha] \cdot \|u\|_2^2$ . (2p)

**Ans.**

- (a) We must have  $w'u' + wu'' = wu'' + \alpha wu'$  from which it follows that  $w' = \alpha w$ , or  $w(x) = e^{\alpha x}$ .  
 (b) For the log norm, we have

$$\langle u, u'' + \alpha u' \rangle = \langle u, u'' \rangle + \alpha \langle u, u' \rangle = \langle u, u'' \rangle,$$

as  $\langle u, u' \rangle = 0$  for all differentiable functions satisfying homogeneous boundary conditions. Further, using integration by parts,  $\langle u, u'' \rangle = -\langle u', u' \rangle \leq -\pi^2 \|u\|_2^2$ , where the last inequality follows from Sobolev's lemma. Hence

$$\mu_2[\mathcal{L}_\alpha] = -\pi^2,$$

independent of  $\alpha$ .

7. **(4p)** Consider the convection–diffusion equation

$$u_t = u_{xx} + \alpha u_x$$

with homogeneous boundary conditions, and initial condition  $u(0, x) = g(x)$ .

- (a) Assuming that you use the same discretization as in the eigenvalue problem for the operator  $\mathcal{L}_\alpha$  above, what is the CFL condition on  $\Delta t$  in order to guarantee stability when the Explicit Euler method is used as the time stepping method? **Hint:** Use the expression for the eigenvalues of a non-symmetric tridiagonal Toeplitz matrix given above. (2p)

- (b) Propose an implicit time stepping method of your choice, so that the CFL condition is overcome, and construct the linear system of equations that needs to be solved in each time step. (2p)

**Ans.**

- (a) The method of lines system is  $\dot{u} = Au$ , with the  $N \times N$  matrix (using  $\Delta x = 1/(N + 1)$ )

$$A = \frac{1}{\Delta x^2} \begin{pmatrix} -2 & 1 + \alpha\Delta x/2 & & & \\ 1 - \alpha\Delta x/2 & -2 & 1 + \alpha\Delta x/2 & & \\ & & \ddots & & \\ & & & & 1 - \alpha\Delta x/2 & -2 \end{pmatrix}$$

As its eigenvalues are

$$\lambda_k[A] = -\frac{2}{\Delta x^2} + \frac{2}{\Delta x^2} \sqrt{1 - \frac{\alpha^2 \Delta x^2}{4}} \cos \frac{k\pi}{N+1}$$

the largest negative eigenvalue is, approximately,

$$\lambda_N[A] \approx -\frac{2}{\Delta x^2} \left( 1 + \sqrt{1 - \frac{\alpha^2 \Delta x^2}{4}} \right)$$

As the CFL condition for the explicit Euler method demands  $|\Delta t \cdot \lambda_N[A]| \leq 2$ , we get

$$\frac{\Delta t}{\Delta x^2} \left( 1 + \sqrt{1 - \frac{\alpha^2 \Delta x^2}{4}} \right) \leq 1,$$

given that the mesh Péclet condition  $|\alpha|\Delta x \leq 2$  is fulfilled.

- (b) By instead choosing (let's say) the Implicit Euler method, we have to solve the linear system

$$(I - \Delta t \cdot A)u_{n+1} = u_n$$

on each step. Because the method is A-stable, there is no longer any restriction on the time step  $\Delta t$ . The method is *unconditionally stable*.

GOOD LUCK!  
G.S.