Numerical Methods for Differential Equations FMNN10/NUMN12
Exam 2013-12-16 **Results to be announced 2013-12-23**
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Exam duration 14:00 – 19:00. A minimum of 16 points out of 32 are required to pass. Your grade is determined by the sum of your exam and project scores, in accordance with the rules on the course home page.

No computers, pocket calculators, cell phones, browsing tablets or any other electronic devices, and no textbooks, lecture notes or written material, may be used during the exam.

1. **(5p)** If the initial value problem $\dot{y} = f(y)$ is solved by the explicit midpoint method, with initial conditions $y_0 = y(0)$ and y_1, we have the recursion

$$y_{n+1} - y_{n-1} = 2hf(y_n).$$

   (a) Determine the order of consistency of the method. (2p)

   (b) Apply the method to the linear test equation $\dot{y} = \lambda y$ and determine the method's stability region. (3p)

Solution

We try the formula for polynomials $P(t) = t^m$. For simplicity, take $t_{n-1} = -h$; $t_n = 0$; $t_{n+1} = h$. Note that if $y(t) = t^m$, then $\dot{y} = f(y) = mt^{m-1}$, for $m \geq 1$. Moreover, the left hand side is always zero for m odd.

The formula obviously holds for $m = 0$ (because the sum of the coefficients in the left hand side is zero). For $m = 1$ we get

$$h - (-h) = 2h \cdot 1$$

which is obviously true for all h. For $m > 1$ we get

$$h^m - (-h)^m = 2h \cdot 0,$$

which holds for m even, i.e., for $m = 2$, but it does not hold for $m = 3$. Hence the order of consistency is $p = 2$.

Applying the method to $\dot{y} = \lambda y$, we get

$$y_{n+1} - y_{n-1} = 2h\lambda y_n$$

with characteristic equation

$$r^2 - 2h\lambda r - 1 = 0.$$

The product of the two roots is -1, so if one root is denoted by r the other root is $-1/r$. Note that if one of the roots is less that 1 in magnitude, the other one is greater than 1 in magnitude. Therefore, in order to have stability, we must have $r = \mathrm{e}^{\mathrm{i}\varphi}$ for some φ. As the sum of the two roots is $2h\lambda$, we get

$$\frac{\mathrm{e}^{\mathrm{i}\varphi} - \mathrm{e}^{-\mathrm{i}\varphi}}{2} = h\lambda,$$

or simply $h\lambda = \mathrm{i}\sin\varphi$. The stability region, therefore, is the (open) interval $(-\mathrm{i}, \mathrm{i})$. The endpoints are excluded as the characteristic equation then has a double root.

2. **(5p)** Consider an implicit Runge-Kutta method with Butcher tableau

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/3 & 2/3 \\ \hline b^{\mathrm{T}} & 1/3 & 2/3 \end{array}$$

   (a) Apply the method to the linear test equation $\dot{y} = \lambda y$ with $y(0) = 1$ and find the method's stability function $R(h\lambda)$. (3p)

   (b) Is the method A-stable? (2p)

Solution

The method can be written

$$\begin{aligned} hY_1' &= hf(y_n) \\ hY_2' &= hf(y_n + hY_1'/3 + 2hY_2'/3) \end{aligned}$$

with updating formula

$$y_{n+1} = y_n + \frac{hY_1'}{3} + \frac{2hY_2'}{3}.$$

Note that the second stage derivative equation is $hY_2' = hf(y_{n+1})$.

When we apply this method to the test equation $y' = \lambda y$ with $y(0) = 1$, we get

$$\begin{aligned} hY_1' &= h\lambda \\ hY_2' &= h\lambda(1 + h\lambda/3 + 2hY_2'/3) \end{aligned}$$

Because the method is implicit, we have to solve for the stage derivative hY_2'. But we also note that, by the updating formula, $hY_2' = h\lambda y_1$. Thus

$$(1 - 2h\lambda/3)hY_2' = h\lambda(1 + h\lambda/3),$$

so

$$y_1 = \frac{1 + h\lambda/3}{1 - 2h\lambda/3} = R(h\lambda).$$

To check whether the method is A-stable, we note that the pole of $R(h\lambda)$ is $h\lambda = 3/2$, hence in the right half plane. The modulus of $R(h\lambda)$ on the imaginary axis is

$$|R(hi\omega)|^2 = \frac{1 + h^2\omega^2/9}{1 + 4h^2\omega^2/9} \le 1,$$

so the method is indeed A-stable.

3. **(4p)** Consider the following nonlinear two-point boundary value problem:

$$\begin{aligned} y'' - (y^2)' + y &= g(x) \\ y(0) = \alpha, \qquad y(1) &= \beta. \end{aligned}$$

   (a) Introduce a suitable grid and discretize with a standard second order method. Give all details about the grid (number of grid points and their location, as well as mesh width Δx) and formulate the discretization. Include the boundary conditions in the equation system. (2p)

   (b) Construct the Jacobian matrix associated with the system. (2p)

Solution

We solve this problem by using the standard 2nd order approximation where we introduce an equidistant grid with N interior points $x_j = j\Delta x$ and $\Delta x = 1/(N+1)$. The discretization is

$$\begin{aligned} \frac{\alpha - 2y_1 + y_2}{\Delta x^2} - \frac{-\alpha^2 + y_2^2}{2\Delta x} + y_1 &= g(x_1) \\ \frac{y_{j-1} - 2y_j + y_{j+1}}{\Delta x^2} - \frac{-y_{j-1}^2 + y_{j+1}^2}{2\Delta x} + y_j &= g(x_j) \\ \frac{y_{N-1} - 2y_N + \beta}{\Delta x^2} - \frac{-y_{N-1}^2 + \beta^2}{2\Delta x} + y_N &= g(x_N). \end{aligned}$$

Retaining the unknowns in the left hand side, and moving the known data to the right hand side, we get

$$\begin{aligned} \frac{-2y_1 + y_2}{\Delta x^2} - \frac{y_2^2}{2\Delta x} + y_1 &= g(x_1) - \frac{\alpha}{\Delta x^2} - \frac{\alpha^2}{2\Delta x} \\ \frac{y_{j-1} - 2y_j + y_{j+1}}{\Delta x^2} - \frac{-y_{j-1}^2 + y_{j+1}^2}{2\Delta x} + y_j &= g(x_j) \\ \frac{y_{N-1} - 2y_N}{\Delta x^2} - \frac{-y_{N-1}^2}{2\Delta x} + y_N &= g(x_N) - \frac{\beta}{\Delta x^2} + \frac{\beta^2}{2\Delta x}. \end{aligned}$$

The Jacobian matrix becomes

$$J(y) = \frac{1}{\Delta x^2} \begin{pmatrix} -2+\Delta x^2 & 1-y_2\Delta x & & \\ 1+y_1\Delta x & -2+\Delta x^2 & 1-y_3\Delta x & \\ & & \ddots & \\ & & 1+y_{N-1}\Delta x & -2+\Delta x^2 \end{pmatrix}$$

4. **(5p)** Two students, **A** and **B**, want to solve the eigenvalue problem

$$u'' + u = \lambda u$$

with one Dirichlet boundary condition $u(0) = 0$ and one Robin condition $u(1) + u'(1) = 0$ on $[0, 1]$.

Student A decides to introduce a grid with $x_j = j\Delta x$, taking $\Delta x = 1/(N + 1/2)$, and then approximates the Robin condition by

$$\frac{u_{N+1} + u_N}{2} + \frac{u_{N+1} - u_N}{\Delta x} = 0.$$

Solving for u_{N+1}, student **A** gets

$$u_{N+1} = \frac{2 - \Delta x}{2 + \Delta x} u_N$$

and proceeds to construct a second order method, based on an $N \times N$ tridiagonal matrix A. Because **A** uses the discretization

$$\frac{u_{j-1} - 2u_j + u_{j+1}}{\Delta x^2} + u_j = \lambda_{\Delta x} u_j,$$

the *last row* of the matrix A, which accounts for the Robin condition, reads

$$\begin{pmatrix} 0 \dots 0 & \frac{1}{\Delta x^2} & \frac{-2 - 3\Delta x + 2\Delta x^2 + \Delta x^3}{\Delta x^2(2 + \Delta x)} \end{pmatrix}$$

Student B, however, insists that one can take a grid with $x_j = j\Delta x$, taking $\Delta x = 1/(N + 1)$ so that $x_{N+1} = 1$, and still obtain a second order method. **B** approximates the derivative at $x = 1$ by the second order BDF2 method:

$$u'(1) \approx \frac{1}{\Delta x} \left(\frac{3}{2} u_{N+1} - 2u_N + \frac{1}{2} u_{N-1} \right).$$

Student **B** then proceeds with exactly the same discretization as **A**, except taking $\Delta x = 1/(N + 1)$ and representing the Robin boundary condition using the BDF2 approximation. **B** constructs a linear algebraic eigenvalue problem $Bu = \lambda u$.

(a) Show that student **B**'s BDF2 approximation is second order accurate, i.e., that

$$u'(1) = \frac{1}{\Delta x}\left(\frac{3}{2}u_{N+1} - 2u_N + \frac{1}{2}u_{N-1}\right) + \mathrm{O}(\Delta x^2).$$

(2p)

(b) In the approximation of the Robin condition $u(1) + u'(1) = 0$, solve for u_{N+1} in terms of u_N and u_{N-1}. (1p)

(c) Construct the last row of student **B**'s matrix B. (2p)

Solution We take $u_{N+1} = u(1)$, with $u_N = u(1-\Delta x)$ and $u_{N-1} = u(1-2\Delta x)$ and expand in Taylor series around $x = 1$. Thus we find

$$\begin{aligned} u_N &= u(1) - \Delta x u'(1) + \frac{\Delta x^2}{2}u''(1) - \frac{\Delta x^3}{6}u'''(1) + \dots \\ u_{N-1} &= u(1) - 2\Delta x u'(1) + \frac{4\Delta x^2}{2}u''(1) - \frac{8\Delta x^3}{6}u'''(1) + \dots \end{aligned}$$

Therefore, given that $u_{N+1} = u(1)$, we have

$$\frac{3}{2}u_{N+1} - 2u_N + \frac{1}{2}u_{N-1} = \Delta x u'(1) - \frac{2\Delta x^3}{6}u'''(1) + \dots,$$

so we have obtained the second order approximation

$$\frac{1}{\Delta x}\left(\frac{3}{2}u_{N+1} - 2u_N + \frac{1}{2}u_{N-1}\right) = u'(1) + \mathrm{O}(\Delta x^2).$$

Student **B** then approximates the Robin condition

$$u(1) + u'(1) \approx u_{N+1} + \frac{1}{\Delta x}\left(\frac{3}{2}u_{N+1} - 2u_N + \frac{1}{2}u_{N-1}\right) = 0,$$

and solves for u_{N+1} to find

$$(2\Delta x + 3)u_{N+1} = 4u_N - u_{N-1},$$

i.e.,

$$u_{N+1} = \frac{4u_N}{3 + 2\Delta x} - \frac{u_{N-1}}{3 + 2\Delta x}.$$

This expression is then substituted into the last equation, in order to account for the boundary condition at $x = 1$. The last equation becomes

$$\frac{u_{N-1} - 2u_N + \frac{4u_N}{3+2\Delta x} - \frac{u_{N-1}}{3+2\Delta x}}{\Delta x^2} + u_N = \lambda_{\Delta x} u_N,$$

so the last row of the matrix B, which accounts for the Robin condition, reads

$$\begin{pmatrix} 0 \dots 0 & \frac{2+2\Delta x}{\Delta x^2(3+2\Delta x)} & \frac{-2-4\Delta x+3\Delta x^2+2\Delta x^3}{\Delta x^2(3+2\Delta x)} \end{pmatrix}$$

5. **(5p)** For $t \geq 0$ and $x \in [0,1]$, let u_j^n approximate $u(j \cdot \Delta x, n \cdot \Delta t)$. Write down the *differential equations* corresponding to the explicit finite difference discretizations below, and give *the name of the equation* in each case.

   (a)
   $$\frac{u_j^{n+1}-u_j^n}{\Delta t} + u_j^n \cdot \frac{u_{j+1}^n - u_j^n}{\Delta x} = 0$$

   (b)
   $$\frac{u_j^{n+1}-u_j^n}{\Delta t} = \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2} + f(u_j^n)$$

   (c)
   $$\frac{u_j^{n+1}-2u_j^n+u_j^{n-1}}{\Delta t^2} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

   (d)
   $$\frac{u_j^{n+1}-u_j^n}{\Delta t} + \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - \Delta t \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2\Delta x^2} = 0$$

   (e)
   $$\frac{u_j^{n+1}-u_j^n}{\Delta t} = \frac{u_{j+1}^n - u_j^n}{\Delta x} + \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2}$$

   Solution

   The equations are: **(a)** the *inviscid Burgers equation* $u_t + uu_x = 0$; **(b)** the *reaction–diffusion equation* $u_t = u_{xx} + f(u)$; **(c)** the *wave equation* $u_{tt} = u_{xx}$; **(d)** the *advection equation* $u_t + u_x = 0$, discretized by the Lax–Wendroff method; and **(e)** the *convection–diffusion equation* $u_t = u_x + u_{xx}$.

6. **(4p)**

   (a) Show that any differentiable function $u(x)$ on $[0,1]$, satisfying the "periodic boundary condition" $u(0) = u(1)$, has the property $\langle u, u' \rangle = 0$, where

   $$\langle u, v \rangle = \int_0^1 u(x)v(x)\,\mathrm{d}x.$$

   (2p)

(b) Conclude that in the advection equation $u_t = u_x$ with periodic boundary conditions $u(t,0) = u(t,1)$, the norm $\|u(t,\cdot)\|_2$ of the solution remains constant for all t. Here $\|u(t,\cdot)\|_2^2 = \langle u, u\rangle$. (2p)

Solution

Integration by parts shows that

$$\langle u, u'\rangle = \int_0^1 uu' \, \mathrm{d}x = [u^2]_0^1 - \int_0^1 u'u \, \mathrm{d}x.$$

Because $u^2(1) = u^2(0)$, we have $\langle u, u'\rangle = -\langle u, u'\rangle$. Hence $\langle u, u'\rangle = 0$. For the advection equation, note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t,\cdot)\|_2^2 = 2\langle u, u_t\rangle = 2\langle u, u_x\rangle = 0.$$

Therefore $\|u(t,\cdot)\|_2$ remains constant, equal to $\|u(0,\cdot)\|_2$.

7. **(4p)** Consider the convection–diffusion equation

$$u_t = u_{xx} + u_x$$

with homogeneous boundary conditions, and initial condition $u(0,x) = g(x)$.

(a) Introduce a suitable notation and write down a standard 2nd order method-of-lines discretization in space combined with the *trapezoidal rule* for time-stepping ("Crank–Nicolson's method"). Give the details about your grid and choice of Δx. (2p)

(b) As the method is implicit, one will have to solve a linear system of equations on each step. Construct this system. (2p)

Solution

We take $x_j = j \cdot \Delta x$ with $\Delta x = 1/(N+1)$, so that there are N internal points, resulting in a method-of-lines (MOL) discretization having N equations. Introduce the Toeplitz matrices

$$S_{\Delta x} = \frac{1}{2\Delta x}\begin{pmatrix} 0 & 1 & & \\ -1 & 0 & 1 & \\ & & \ddots & \\ & & -1 & 0 \end{pmatrix} \qquad T_{\Delta x} = \frac{1}{\Delta x^2}\begin{pmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & & \ddots & \\ & & 1 & -2 \end{pmatrix}$$

to approximate $\mathrm{d}/\mathrm{d}x$ and $\mathrm{d}^2/\mathrm{d}x^2$, respectively, on the grid. With $u_j(t) \approx u(x_j, t)$ we get the MOL equation

$$\dot{u} = (T_{\Delta x} + S_{\Delta x})u.$$

Applying the trapezoidal rule to this ODE gives

$$u^{n+1} = u^n + \frac{\Delta t}{2}\left((T_{\Delta x} + S_{\Delta x})u^n + (T_{\Delta x} + S_{\Delta x})u^{n+1}\right),$$

where the superscript refers to the time step index and $u_j^n \approx u(x_j, t^n)$. As the method is implicit, we have to solve for u^{n+1} from

$$\left(I - \frac{\Delta t}{2}(T_{\Delta x} + S_{\Delta x})\right)u^{n+1} = \left(I + \frac{\Delta t}{2}(T_{\Delta x} + S_{\Delta x})\right)u^n.$$

The matrix is tridiagonal, implying that solving the system is not time-consuming.

Lycka till!
G.S.