

Numerical Methods for Differential Equations exam 2017-08-21  
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Exam duration 08:00 – 12:00. Place: Vic:2B. A minimum of 16 points out of 32 are required to pass. Your grade is determined by the sum of your exam and project scores, in accordance with the rules published on the course home page.

No computers, pocket calculators, cell phones, browsing tablets or any other electronic devices, and no textbooks, lecture notes or written material, may be used during the exam. Only paper, pen or pencil, and personal competence, are permitted.

1. (6p) For the initial value problem  $\dot{y} = f(y)$  the following method is proposed, in backward difference notation,

$$\left(\nabla + \frac{\nabla^2}{2}\right) y_n = \left(1 - \frac{\nabla^2}{3}\right) hf(y_n).$$

- (a) Find the method's order of consistency. (3p)  
 (b) Apply the method to the linear test equation  $\dot{y} = \lambda y$ , and determine whether the method is A-stable or not. (2p)  
 (c) Construct the nonlinear equation that has to be solved on each step to use the method. (1p)
2. (5p) Consider the 3-stage embedded Runge-Kutta pair of orders  $p = 2$  and  $p = 3$  given by the Butcher tableau

0	0	0	0
1/2	1/2	0	0
3/4	0	3/4	0
$M_1$	0	1	0
$M_2$	2/9	3/9	4/9

where " $M_1$ " refers to the first method and " $M_2$ " to the second.

- (a) Write the methods in terms of stage values, stage derivatives and evaluations of the function  $f$ , when they are applied to the problem  $\dot{y} = f(y)$ . (2p)  
 (b) Which method is of order 2 and which is of order 3? (1p)  
 (c) The embedded pair is used as follows: starting at a point  $y_n$ , method  $M_1$  produces a result  $u_{n+1} = M_1(y_n)$ , while method  $M_2$  produces a result  $v_{n+1} = M_2(y_n)$ . The difference

$$u_{n+1} - v_{n+1} = (M_1 - M_2)(y_n)$$

is used to estimate the error in a single step. Find an expression for this error estimate when the method is applied to the linear test equation  $\dot{y} = \lambda y$ . (2p)

3. (4p) Construct a second order discretization of the nonlinear two-point boundary value problem

$$\begin{aligned} u'' + (u^2)' - u &= f(x) \\ u(0) &= 0, \quad u'(1) = 1. \end{aligned}$$

- (a) Introduce a grid (draw a picture) and discretize with a standard symmetric second order finite difference method. Be careful to explain your notation, and how  $\Delta x$  is related to the number of equations,  $N$ . Make sure to write down the first and the last equations clearly. Make sure to write down the first and the last equations clearly. (2p)
- (b) As the system is nonlinear we need to solve it using Newton's method. Find the first and last rows of the Jacobian matrix of the system. (2p)
4. (4p) The following Sturm–Liouville eigenvalue problem is given:

$$\frac{d}{dx} \left( a(x) \frac{du}{dx} \right) = \lambda u$$

with Dirichlet boundary conditions  $u(0) = u(1) = 0$ . Construct a second order discretization of this problem to formulate it as a linear algebraic eigenvalue problem. Include all important details on grid point locations, mesh size  $\Delta x$ , boundary conditions etc., and write the system in matrix–vector form.

5. (5p) Let  $u(x)$  be differentiable on  $[0, 1]$  with periodic boundary conditions  $u(0) = u(1)$ , and let the inner product  $\langle \cdot, \cdot \rangle$  be defined by

$$\langle u, v \rangle = \int_0^1 uv \, dx.$$

- (a) Show that  $u'$  is orthogonal to  $u^2$ . (2p)
- (b) Consider the inviscid Burgers equation

$$u_t - \left( \frac{u^2}{2} \right)_x = 0$$

with periodic boundary conditions and assume that the solution is continuously differentiable. Show that  $\|u(t, \cdot)\|_2$  remains constant for all  $t$ . (3p)

6. (5p) For  $t \geq 0$  and  $x \in [0, 1]$ , write down the following PDEs

- (a) Advection equation
- (b) Reaction-diffusion equation
- (c) Convection-diffusion equation
- (d) Viscous Burgers equation
- (e) Wave equation

For each equation, state whether the problem is *elliptic*, *parabolic* or *hyperbolic*, and state the expected structure of the CFL condition when you use an explicit time-stepping procedure such as the explicit Euler method.

7. (3p) The Crank-Nicolson method for the linear diffusion equation  $u_t = u_{xx}$  with homogeneous Dirichlet boundary conditions is obtained by combining a standard 2nd order method-of-lines discretization in space with the trapezoidal rule for time-stepping.

- (a) Introduce a suitable notation and write down this method as a vector-matrix recursion. (2p)
- (b) Using the well-known eigenvalues of the  $N \times N$  Toeplitz matrix,

$$\lambda_k[T_{\Delta x}] = -4 \cdot (N + 1)^2 \cdot \sin^2 \frac{k\pi}{2(N + 1)} \quad k = 1, \dots, N.$$

show that the Crank-Nicolson method is *unconditionally stable* for the diffusion equation, i.e., there is no CFL condition and the method is stable for every  $\Delta t > 0$ . (1p)

LYCKA TILL — GOOD LUCK! G.S.