

Exam duration 08:00 – 12:00. A minimum of 16 points out of 32 are required to pass. Your grade is determined by the sum of your exam and project scores, in accordance with the rules published on the course home page.

No computers, pocket calculators, cell phones, browsing tablets or any other electronic devices, and no textbooks, lecture notes or written material, may be used during the exam.

1. (5p) A two-step linear multistep method has the form

$$y_{n+2} - y_{n+1} = h(\beta_2 f(y_{n+2}) + \beta_1 f(y_{n+1}) + \beta_0 f(y_n))$$

for the initial value problem

$$\dot{y} = f(y); \quad y(0) = y_0.$$

- (a) Determine the coefficients $\beta_0, \beta_1, \beta_2$ so that the consistency order is maximal. (3p)
- (b) If $\beta_2 = 0$ the method is explicit. How does this affect the maximal order and why? (2p)
2. (5p) Consider the following 2-stage Runge-Kutta method,

$$\begin{aligned} hY_1' &= hf(y_n) \\ hY_2' &= hf(y_n + (hY_1' + hY_2')/2) \\ y_{n+1} &= y_n + (hY_1' + hY_2')/2 \end{aligned}$$

applied to the problem $\dot{y} = f(y)$.

- (a) Write down the method in terms of its Butcher tableau. (1p)
- (b) Determine the method's stability function $R(h\lambda)$. (2p)
- (c) Is the method A-stable or not? (Demonstrate why.) (2p)
3. (5p) The following linear two-point boundary value problem is given:

$$\begin{aligned} u'' + x^2 u' - u &= x^2 \\ u'(0) &= 1; \quad u(1) = 1. \end{aligned}$$

- (a) Introduce a grid and discretize with a standard *second order* finite difference method. Be careful to define Δx , write down all equations, and show exactly how the boundary conditions affect the system by writing down the first and last equations separately. (3p)
- (b) This results in a linear system of equations, $L_N u = f$. Construct the matrix L_N . (2p)

4. (5p) Consider the following two-point boundary value problem:

$$\begin{aligned} \epsilon u'' + u' - u &= g(x) \\ u(0) &= 0, \quad u(1) = 0, \end{aligned}$$

where $0 < \epsilon \ll 1$ is a real constant. The solvability of this problem depends on ellipticity. This is governed by the properties of the operator

$$\mathcal{L} = \epsilon \frac{d^2}{dx^2} + \frac{d}{dx} - 1.$$

- (a) Use integration by parts to find the *logarithmic norm* $\mu_2[\mathcal{L}]$. (3p)
- (b) How small can we take ϵ without losing ellipticity, if we interpret ellipticity to mean that $\mu_2[\mathcal{L}] < 0$? (2p)
5. (5p) Consider the following PDEs for $t \geq 0$ and $x, y \in [0, 1]$ together with suitable boundary and initial conditions:

- (a) $u_t + uu_x = 0$
- (b) $u_t + uu_x = u_{xx}$
- (c) $u_t = \epsilon u_{xx} + f(u)$
- (d) $u_t = a(x)u_x$
- (e) $u_{xx} + u_{yy} = 0$

For each equation, classify the problem as *elliptic*, *parabolic* or *hyperbolic*. In addition, give the *name* of each equation, or, in case it has no name, name it based on the terms that enter the equation.

6. (7p) Consider the linear advection equation, $u_t = u_x$, with periodic boundary conditions $u(t, 0) = u(t, 1)$ on $[0, 1]$ and with initial condition $u(0, x) = g(x)$. Let us use the Lax–Friedrichs method to solve the problem, using the full discretization

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^n + u_{j-1}^n}{2} + \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

- (a) If we let u^n denote the vector $(u_1^n, u_2^n, \dots, u_N^n)^T$, the Lax–Friedrichs method can be written

$$u^{n+1} = (I + \Delta t A + \frac{\Delta t}{\Delta x} B) u^n.$$

Construct the $N \times N$ matrices A and B . (2p)

- (b) Classify the matrices A and B as symmetric, skew-symmetric, unsymmetric or circulant. Finally, classify the total matrix $I + \Delta t A + \frac{\Delta t}{\Delta x} B$, in the same way. (3p)
- (c) Instead of using the Lax–Friedrichs time-stepping, consider the trapezoidal rule

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x} + \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}.$$

What are the advantages and one drawbacks of using this method instead of the previous method? What happens to the CFL condition, and what is the CFL condition for the Lax–Friedrichs method? (2p)

LYCKA TILL — GOOD LUCK! G.S.