Numerical Methods for Differential Equations exam **2016-05-12**
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Exam duration 08:00 – 12:00. A minimum of 16 points out of 32 are required to pass. Your grade is determined by the sum of your exam and project scores, in accordance with the rules published on the course home page.

No computers, pocket calculators, cell phones, browsing tablets or any other electronic devices, and no textbooks, lecture notes or written material, may be used during the exam.

1. **(5p)** The two-step Adams-Moulton method has the form

$$y_{n+2} - y_{n+1} = h\left(\beta_2 f(y_{n+2}) + \beta_1 f(y_{n+1}) + \beta_0 f(y_n)\right)$$

for the initial value problem

$$\dot{y} = f(y); \qquad y(0) = y_0.$$

   (a) Determine the coefficients $\beta_0, \beta_1, \beta_2$ so that the consistency order is $p = 3$. (3p)

   (b) Is the resulting method *zero-stable*? (Motivate your answer.) (1p)

   (c) Is the resulting method *A-stable*? (Motivate your answer.) (1p)

2. **(5p)** Consider the 2-stage Runge-Kutta method given by the Butcher tableau

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}$$

   (a) Write the method in terms of *stage values and stage derivatives*, when it is applied to the problem $\dot{y} = f(y)$. (1p)

   (b) Determine the method's stability function $R(h\lambda)$. (2p)

   (c) Show that the method is A-stable. (2p)

3. **(5p)** The following linear two-point boundary value problem is given:

$$u'' + xu' - u = f(x)$$
$$u(0) = 1; \quad u(1) + u'(1) = 1.$$

(a) Introduce a grid and discretize with a standard *second order* finite difference method. Be careful to define Δx, write down all equations, and show exactly how the boundary conditions affect the system by writing down the first and last equations separately. (3p)

(b) This results in a linear system of equations, $L_N u = f$. Construct the matrix L_N. (2p)

4. **(5p)** Consider the following two-point boundary value problem:

$$u'' + u' + Ku = -g(x)$$
$$u(0) = 0, \qquad u(1) = 0,$$

where K is a real constant. The solvability of this problem depends on whether the problem is elliptic or not. This is governed by the properties of the operator

$$\mathcal{L} = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}}{\mathrm{d}x} + K,$$

which obviously depends on K.

(a) Use integration by parts to find the *logarithmic norm* $\mu_2[\mathcal{L}]$. (3p)

(b) Find out exactly for what values of K we have $\mu_2[\mathcal{L}] < 0$ and explain what happens if $K = \pi^2$. (2p)

5. **(5p)** Consider the following PDEs for $t \geq 0$ and $x, y \in [0, 1]$:

(a) $u_t + au_x = 0$

(b) $-u_{xx} - u_{yy} = f(x, y)$

(c) $u_t = \epsilon u_{xx} + u_x$

(d) $u_t + uu_x = u_{xx}$

(e) $u_t = \frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right)$

For each equation, classify the problem as *elliptic, parabolic or hyperbolic*. In addition, give the *name* of each equation, or, in case it has no name, name it based on the terms that enter the equation.

6. **(7p)** Consider the linear advection equation, $u_t = u_x$, with periodic boundary conditions $u(t, 0) = u(t, 1)$ on $[0, 1]$ and with initial condition $u(0, x) = g(x)$. Let us use the Lax–Friedrichs method to solve the problem, using the full discretization

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^n + u_{j-1}^n}{2} + \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

(a) If we let u^n denote the vector $(u_1^n, u_2^n, \ldots, u_N^n)^{\mathrm{T}}$, the Lax–Friedsrichs method can be written

$$u^{n+1} = (I + \Delta t A + \frac{\Delta t}{\Delta x} B) u^n.$$

What are the $N \times N$ matrices A and B? (3p)

(b) Is A symmetric or skew-symmetric? Is B symmetric or skew-symmetric? Motivate your answers. (2p)

(c) What would be the advantage and drawback of instead using the time-stepping scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x} + \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

(2p)

LYCKA TILL — GOOD LUCK! G.S.