Numerical Methods for Differential Equations FMNN10/NUMN12
Exam 2015-05-07, MA08-C

Exam duration 08:00 13:00. A minimum of 16 points out of 32 are required to pass. Your grade is determined by your score on this exam and your project scores from the fall of 2014, in accordance with the rules on the course home page.

No computers, pocket calculators, cell phones, browsing tablets or any other electronic devices, and no textbooks, lecture notes or written material, may be used during the exam.

1. **(6p)** If the diffusion equation $u_t = u_{xx}$ is discretized using the method of lines, one obtains an initial value problem $u' = T_{\Delta x} u$.
   (a) Explain why this problem is considered "stiff." (2p)
   (b) The BDF2 method for an initial value problem $y' = f(y)$ is given by
   $$\frac{3}{2}y_{n+1} - 2y_n + \frac{1}{2}y_{n-1} = hf(y_{n+1}),$$
   where $h = \Delta t$ is the time step size. The method is suitable for stiff equations. Apply it to $u' = T_{\Delta x}u$, and find the linear system of equations that has to be solved on each step. You don't need to specify the matrix $T_{\Delta x}$; it is sufficient to use this notation in the problems below. (2p)
   (c) The eigenvalues of $T_{\Delta x}$ are negative real. Is there a restriction on the time step size Δt such that the linear system above is solvable? (I.e., can the system be singular for any $\Delta t > 0$?) Motivate your answer. (1p)
   (d) The BDF2 method is A-stable. Given what you know about the eigenvalues of $T_{\Delta x}$, is there a restriction (CFL condition) on Δt in order to keep the method stable? (I.e., can the method go unstable for any $\Delta t > 0$?) Motivate your answer. (1p)

2. **(5p)** A Runge–Kutta method has the stability function
   $$R(z) = \frac{1 + z/3}{1 - 2z/3 + z^2/6},$$
   where $z = h\lambda$.
   (a) Is the method implicit or explicit? Motivate your answer. (1p)

   (b) What is the minimum number of stages the method must have in order to have this stability function? (1p)
   (c) Determine whether the method is A-stable. (3p)

3. **(5p)** Consider the nonlinear two-point boundary value problem
   $$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{\mathrm{d}u^2}{\mathrm{d}x} + u = 0$$
   with boundary conditions $u(0) = 0$ and $u'(1) = 0$.
   (a) Introduce a suitable grid and discretize with a *standard second order method*. Give *all details about the grid*, such as the number of grid points and their location, as well as mesh width Δx, and formulate the discretization. Include the boundary conditions in the equation system. (3p)
   (b) As the equation is nonlinear, it is necessary to use Newton's method for solving the equation arising from the discretization. Construct the system's Jacobian matrix. (2p)

4. **(5p)** Let $u(x)$ be a differentiable function on $[0, 1]$ with periodic boundary conditions $u(0) = u(1)$, and let the inner product $\langle\cdot,\cdot\rangle$ be defined by
   $$\langle u, v\rangle = \int_0^1 uv\,\mathrm{d}x.$$
   (a) Show that u' is orthogonal to u^2. (2p)
   (b) Consider the inviscid Burgers equation $u_t - uu_x = 0$ with periodic boundary conditions and assume that the solution is continuously differentiable. Show that $\|u(t,\cdot)\|_2$ remains constant for all t. (3p)

5. **(5p)** Classify the following PDEs for $t \geq 0$ and $x \in [0, 1]$:
   (a) $u_t + f(u)_x = 0$
   (b) $u_t + a \cdot u_x = 0$
   (c) $u_t = (p(x)\,u_x)_x$
   (d) $u_{tt} - u_{xx} = f(t, x)$
   (e) $u_{yy} + u_{xx} = f(x, y)$

For each equation, state whether the problem is *elliptic, parabolic or hyperbolic*. All parameters are supposed to be *positive*. In addition, give the *"name"* of each equation (e.g. "convection diffusion equation" etc.).

6. **(6p)** Consider the convection–diffusion equation

$$u_t = u_{xx} + \gamma u_x$$

with homogeneous boundary conditions, and initial condition $u(0, x) = g(x)$. The differential operator

$$\mathcal{L} = \frac{\partial^2}{\partial x^2} + \gamma \frac{\partial}{\partial x}$$

then has negative real eigenvalues,

$$\lambda_k = -(k\pi)^2 - \frac{\gamma^2}{4}.$$

(a) Introduce a grid and use a symmetric second order discretization in space. Write down your discretization in matrix-vector form. (2p)

(b) Given that the eigenvalues of an non-symmetric tridiagonal $N \times N$ Toeplitz matrix $A = \text{tridiag}(b \quad a \quad c)$ are

$$\lambda_k = a + 2\sqrt{bc}\cos\frac{k\pi}{N+1}; \qquad k = 1 : N,$$

what is the largest value of Δx you can choose so that the discrete problem also has negative real eigenvalues? (2p)

(c) Combine your method of lines discretization with the explicit Euler method for time stepping. What is the CFL condition on Δt in order to guarantee stability? (2p)

Good luck!
G.S.