Numerical Methods for Differential Equations FMNN10/NUMN12
Exam 2014-08-22 **Results to be announced 2014-08-29**

Exam duration 14:00 – 19:00. A minimum of 16 points out of 32 are required to pass. Your grade is determined by the sum of your exam and project scores, in accordance with the rules on the course home page.

No computers, pocket calculators, cell phones, browsing tablets or any other electronic devices, and no textbooks, lecture notes or written material, may be used during the exam.

1. **(6p)** The difference corrected BDF methods are claimed to be suitable for nonstiff problems. For the intitial value problem $y' = f(y)$; $y(0) = y_0$ the two-step dc-BDF method reads

$$\frac{3}{2}y_{n+1} - 2y_n + \frac{1}{2}y_{n-1} = \frac{2}{3}hf(y_{n+1}) + \frac{2}{3}hf(y_n) - \frac{1}{3}hf(y_{n-1})$$

   (a) Show that this method is *zero-stable*. (1p)
   (b) Determine the *order of consistency* of the method. (3p)
   (c) Is the resulting method *A-stable*? (2p)

2. **(4p)** Consider the explicit 3-stage Runge-Kutta method

$$\begin{aligned}
hY_1' &= hf(y_n)\\
hY_2' &= hf(y_n + hY_1'/2)\\
hY_3' &= hf(y_n + 3 \cdot hY_2'/4)\\
y_{n+1} &= y_n + (2hY_1' + 3hY_2' + 4hY_3')/9
\end{aligned}$$

   (a) Construct the *Butcher tableau* of this method. (1p)
   (b) Find its *stability polynomial*. (3p)

3. **(4p)** Consider the stationary convection–diffusion–reaction equation

$$\begin{aligned}
y'' + \alpha y' + f(y) &= 0\\
y(0) = 0, \qquad y(1) &= 1.
\end{aligned}$$

   (a) Introduce a suitable grid and discretize with a *standard second order method.* Give *all details about the grid*, such as the number of grid points and their location, as well as mesh width Δx, and formulate the discretization. Include the boundary conditions in the equation system. (2p)

(b) As the problem is nonlinear, it will have to be solved using Newton's method. Construct the Jacobian matrix associated with the discretization. (2p)

4. **(5p)** Consider the eigenvalue problem

$$u'' + u' + u = \lambda u$$

with boundary conditions $u(0) = u(1) = 0$ on $[0, 1]$.

(a) Solve the *analytical eigenvalue* problem and find its eigenvalues λ_k and corresponding eigenfunctions $u_k(x)$. (3p)

(b) Construct an approximating algebraic eigenvalue problem

$$Au = \lambda_{\Delta x} u.$$

by using a *second order discretization* using symmetric difference quotients to approximate the derivatives. Give the matrix A. (2p)

5. **(5p)** Write down the simplest example (using one space dimension) of the following classical PDEs and state whether the problem is *elliptic, parabolic or hyperbolic*:

(a) Reaction-diffusion equation

(b) Advection equation

(c) Convection-diffusion equation

(d) Wave equation

(e) Inviscid Burgers equation

6. **(4p)** In the two-point boundary value problem $u'' + u' + u = f$, the eigenfunctions of the differential operator are of the form

$$u_k(x) = \frac{\sin k\pi x}{\sqrt{w(x)}}, \qquad k = 1, 2, \ldots$$

Introduce the weighted inner product

$$\langle u, v \rangle_w = \int_0^1 u(x)v(x)\, w(x)\mathrm{d}x.$$

(a) Show that the eigenfunctions $u_k(x)$ are *orthogonal* with respect to the weighted inner product $\langle \cdot, \cdot \rangle_w$. (2p)

(b) Compute the corresponding norm $\|u_k\|_w$ of the eigenfunctions. (2p)

7. **(4p)** Consider the nonlinear, parabolic wave equation

$$u_t = \varepsilon u_{xx} + u u_x$$

with homogeneous boundary conditions, and initial condition $u(0, x) = g(x)$. Here ε is a small, positive parameter, $0 < \varepsilon \ll 1$.

(a) Introduce a suitable notation and write down a standard 2nd order method-of-lines discretization in space combined with the explicit Euler method for time-stepping. Give the details about your grid and choice of Δx. (2p)

(b) If instead of the explicit Euler method the implicit Euler is used, one will have to solve a nonlinear system of equations on each step. Construct this system. (2p)

LYCKA TILL!
G.S.