Numerical Methods for Differential Equations FMNN10/NUMN12
Exam 2014-04-29 **Results to be announced 2014-05-06**

Exam duration 14:00 – 19:00. A minimum of 16 points out of 32 are required to pass. Your grade is determined by the sum of your exam and project scores, in accordance with the rules on the course home page.

No computers, pocket calculators, cell phones, browsing tablets or any other electronic devices, and no textbooks, lecture notes or written material, may be used during the exam.

1. **(5p)** You are familiar with the BDF metods, e.g.

$$\frac{3}{2}y_{n+1} - 2y_n + \frac{1}{2}y_{n-1} = hf(y_{n+1})$$

for the initial value problem

$$y' = f(y); \qquad y(0) = y_0.$$

The BDF methods are implicit and approximate y'_{n+1} by a linear combination of y values. But one can also construct *explicit* methods based on backward differences that approximate y'_{n+1}. To the 2-step BDF above there corresponds an explicit method

$$\frac{3}{2}y_{n+1} - 2y_n + \frac{1}{2}y_{n-1} = h\beta_1 f(y_n) + h\beta_0 f(y_{n-1}).$$

   (a) Show that this method is *zero-stable.* (1p)
   (b) Determine the values of the parameters β_0 and β_1 so that the method is of second order. (3p)
   (c) Is the resulting method A-stable? (Motivate your answer.) (1p)

2. **(4p)** Consider the 3-stage Runge-Kutta method with Butcher tableau

0	0	0	0
1/2	1/2	0	0
3/4	0	3/4	0
	2/9	3/9	4/9

   (a) Write down the *formulas* for using this method. (1p)
   (b) Find its *stability function* $R(h\lambda)$ and express it as a rational function, i.e., on the form $P(h\lambda)/Q(h\lambda)$, where P and Q are polynomials. (3p)

3. **(4p)** Consider the stationary convection–diffusion equation

$$y'' + \alpha y' = g(x)$$
$$y(0) = 1, \qquad y(1) = 1.$$

(a) Introduce a suitable grid and discretize with a *standard second order method.* Give *all details about the grid,* such as the number of grid points and their location, as well as mesh width Δx, and formulate the discretization. Include the boundary conditions in the equation system. (2p)

(b) Construct the matrix associated with the discretization. Is it a Toeplitz matrix? Is it symmetric, skew-symmetric, circulant, or of some other structure? (2p)

4. **(6p)** Consider the eigenvalue problem associated with the stationary convection–diffusion operator above, i.e.,

$$u'' + \alpha u' = \lambda u$$

with boundary conditions $u(0) = u(1) = 0$ on $[0, 1]$.

(a) Solve the *analytical eigenvalue* problem and find its eigenvalues λ_k and corresponding eigenfunctions $u_k(x)$. (3p)

(b) Construct an approximating algebraic eigenvalue problem

$$Au = \lambda_{\Delta x} u.$$

by using a *second order discretization* using symmetric difference quotients to approximate the derivatives. Give the matrix A. (1p)

(c) Assuming that $\alpha > 0$, what are the eigenvalues of A if one chooses Δx so that

$$\frac{\alpha \Delta x}{2} = 1\,?$$

Is that result realistic? (2p)

5. **(5p)** Consider the following PDEs for $t \geq 0$ and $x \in [0, 1]$:

(a) $u_t = d \cdot u_{xx} + u_x + f(u)$

(b) $u_t + a \cdot u_x = 0$

(c) $u_t + a \cdot u_x = d \cdot u_{xx}$

(d) $u_{tt} = c^2 \cdot u_{xx}$

(e) $u_t + uu_x = u_{xx}$

For each equation, state whether the problem is *elliptic, parabolic or hyperbolic*, and state *where* (i.e., on which boundary) boundary conditions are required in each case for the problem to be well-posed. All parameters a, c and d are supposed to be *positive* numbers. In addition, give the *"name"* of each equation (e.g. "convection–diffusion equation" etc.).

6. **(4p)** In the stationary convection–diffusion problem $u'' + \alpha u' = f$ studied in the previous problems, the eigenfunctions are of the form

$$u_k(x) = \frac{\sin k\pi x}{\sqrt{w(x)}}, \qquad k = 1, 2, \ldots$$

Introduce the weighted inner product

$$\langle u, v\rangle_w = \int_0^1 u(x)v(x)\, w(x)\mathrm{d}x.$$

   (a) Show that the eigenfunctions $u_k(x)$ are *orthogonal* with respect to the weighted inner product $\langle \cdot, \cdot\rangle_w$. (2p)

   (b) Compute the corresponding norm $\|u_k\|_w$ of the eigenfunctions. (2p)

7. **(4p)** Consider the convection–diffusion equation

$$u_t = u_{xx} + \alpha u_x$$

with homogeneous boundary conditions, and initial condition $u(0, x) = g(x)$.

   (a) Introduce a suitable notation and write down a standard 2nd order method-of-lines discretization in space combined with the *trapezoidal rule* for time-stepping ("Crank–Nicolson's method"). Give the details about your grid and choice of Δx. (2p)

   (b) As the method is implicit, one will have to solve a linear system of equations on each step. Construct this system. (2p)

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G.S.